Math 324 Lecture 5
Dr Bolstad
http://math324sfsu.bmbolstad.com

What is probability?

Probability is the study of randomness and uncertainity. Probability allows us to quantify the chance that a certain event will occur. eg "There is a 30% chance of rain tommorrow?", "Our chance of winning the lottery is 1 in 41,416,353". Probability is based on the idea of a random experiment, that is an experiment where the outcome cannot be predicted with absolute certainty before it is run. In (the frequentist view of) statistics, the probability of an event is the proportion of the time that we observe the event if the random experiment was run an infinite number of times. An alternative view of statistics (called Bayesian statistics) treats probability as our degree of belief that an even will happen. Before we can really get to talking about probabilities and computing probabilities we will first spend some time reviewing some set theory.

Sample space and events

The *sample space* of an experiment is the set of all possible outcomes of that experiment. Mathematically the sample space is denoted by the symbol S. Example, coin tossing provided we assume that it is impossible for a coin to land on its edge, the samplespace is $\{H, T\}$ where H represents heads and T represents tails.

An *event* is any collection (subset) of outcomes contained in the sample space S. We sometimes use the word *simple* to refer to an event which consists of exactly one outcome. The term *compound* is used to represent events that contain more than one outcome. eg toss two coins. Possible outcomes $\{HH, HT, TH, TT\}$. The event "get two heads" has only one outcome in the set of all possible outcomes, HH, and is therefore a simple event. The event "get two of the same side" has two outcomes, HH and TT and so is a compound event.

Some set theory

1. The *union* of two events A and B is denoted by $A \cup B$ and in words can be read "A or B". It is the event which consists of all the outcomes that are either in A alone, or B alone or both A and B.

2. The *intersection* of two events A and B is denoted by $A \cap B$ and in words can be read "A and B". It is the event which consists of all the outcomes that are in both A and B.

3. The *complement* of an event B is the set of all outcomes in S that are not contained in B. Notationally we use B' or B^c or $\bar{B}$ to represent the complement of the event B. Note that $A \cup A^c = S$.

4. Events A and B are *mutually exclusive* if none of the outcomes in A are in B and vice versa. Another term for mutually exclusive is *disjoint*.

5. The event A is a subset of B, denoted $A \subset B$, if every outcome that is a member of A is also a member of B. Note that not every outcome in B need be in A.

Unions and intersections may be extended to more than two events eg $A \cup B \cup C$ is the set of all outcomes that are in at least one of A, B, or C. $A \cap B \cap C$ is the set of all outcomes that is in all of A, B, C. A set of events A, B, C is mutually exclusive if there is no outcome that is a member of any two of A, B, C.

Set Algebra Axioms

1. The set operations are commutative

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

2. The set operations are associative

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

3. There are distributive laws in set algebra

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

4. The symbol for the empty set is $\emptyset$. The empty set contains no outcomes.

$$A \cap \emptyset = \emptyset$$
$$A \cup \emptyset = A$$
$$A \cap A^c = \emptyset$$

5. Some rules involving the universal set S.

$$A \cap S = A$$
$$A \cup S = S$$
$$A \cup A^c = S$$

6. Some other rules about complements

$$
\begin{aligned}
(A^c)^c &= A \\
\emptyset^c &= S \\
S^c &= \emptyset
\end{aligned}
$$

7. Set operations are closed ie If $A \subset S$ and $B \subset S$ then $(A \cap B) \subset S$ and $(A \cup B) \subset S$.

Set Algebra Theorems

Theorem 1

$$
\begin{aligned}
A \cap A &= A \\
A \cup A &= A
\end{aligned}
$$

Theorem 2.

$$
\begin{aligned}
A \cap (A \cup B) &= A \\
A \cup (A \cap B) &= A
\end{aligned}
$$

Theorem 3. DeMorgans Laws for sets

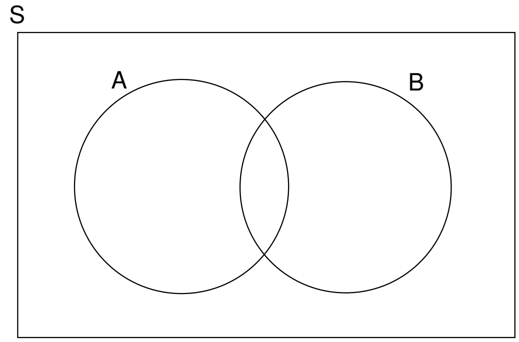
$$
\begin{aligned}
(A \cup B)^c &= A^c \cap B^c \\
(A \cap B)^c &= A^c \cup B^c
\end{aligned}
$$

Another set operation

The set difference $A - B$ sometimes written as $A \backslash B$ is the set of all outcomes in A that are not members of B. eg suppose $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ then $A - B = \{1\}$.

Venn diagrams

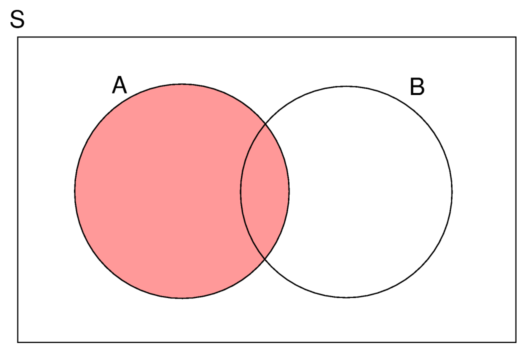
A Venn diagram is a useful way to graphically illustrates sets and the relationships between them. First, a large box is draw which represents the sample space. Circles (or other shapes are) drawn inside represent the different sets. Specifically, outcomes inside the circles are members of the specific sets. We often use shading to represent specific areas on the diagram.



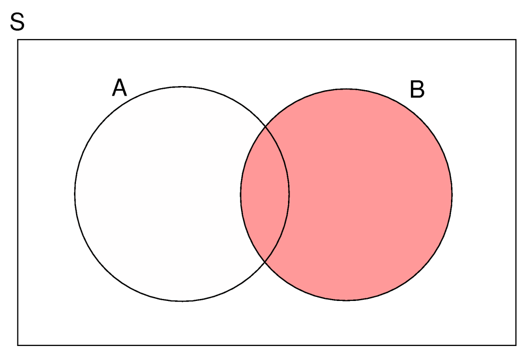
Some Examples

1. Draw Venn diagrams highlighting the following areas

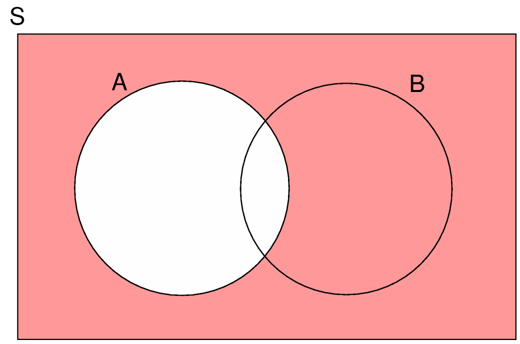
A



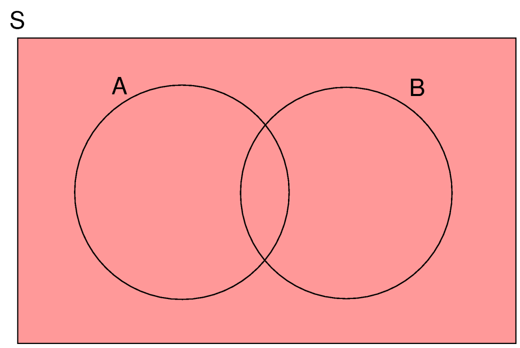
B



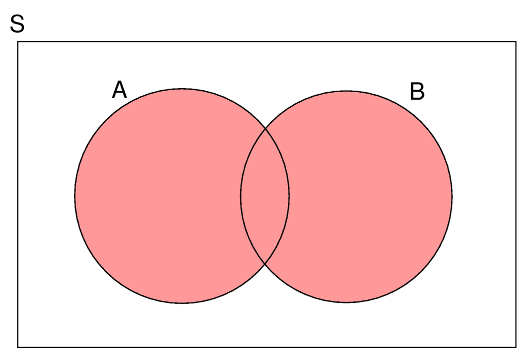
A^c



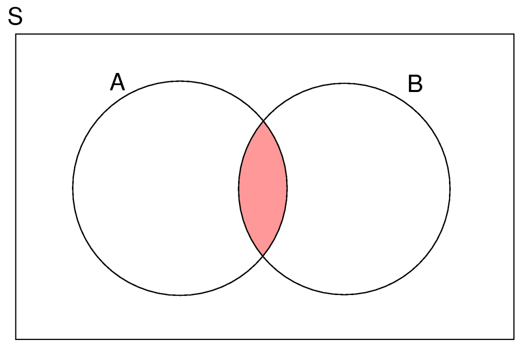
S



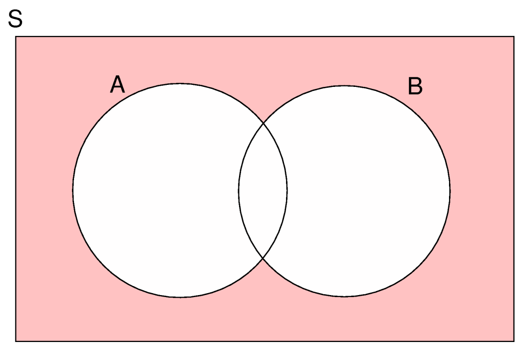
$A \cup B$



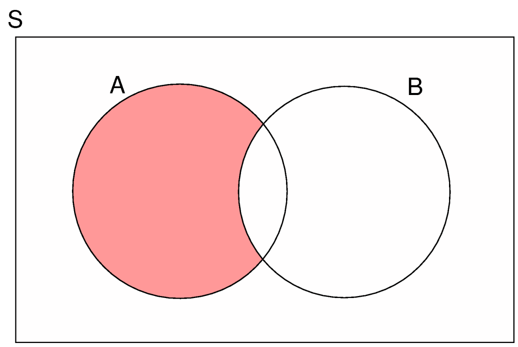
$A \cap B$



$(A \cup B)^c$

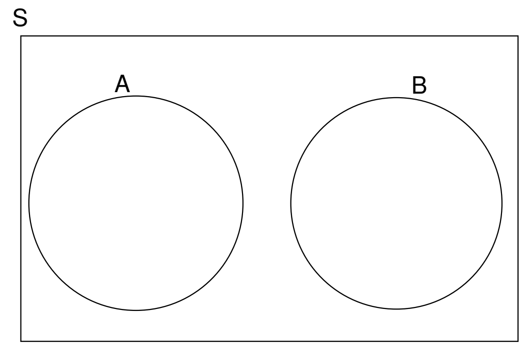


$A \backslash B$

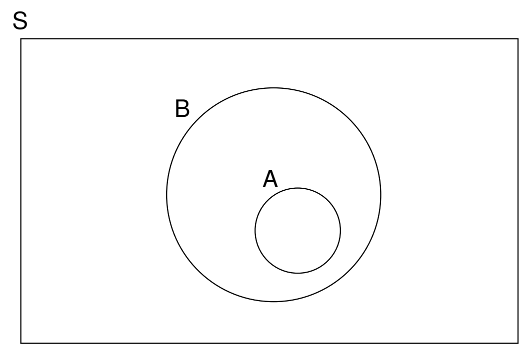


2. Draw Venn Diagrams

   1. With two mutually exclusive sets A and B



   2. With $A \subset B$



3. Prove theorems 1 and 2.

   1. Theorem 1

$$
\begin{aligned}
\text{LHS} &= A \cap A \\
&= (A \cap A) \cup \emptyset \text{ From Axiom 4} \\
&= (A \cap A) \cup (A \cap A^c) \ \text{ From Axiom 4} \\
&= A \cap (A \cup A^c) \ \text{ From Axiom 3} \\
&= A \cap S \text{ From Axiom 5} \\
&= A \text{ From Axiom 5} \\
&= \text{RHS}
\end{aligned}
$$

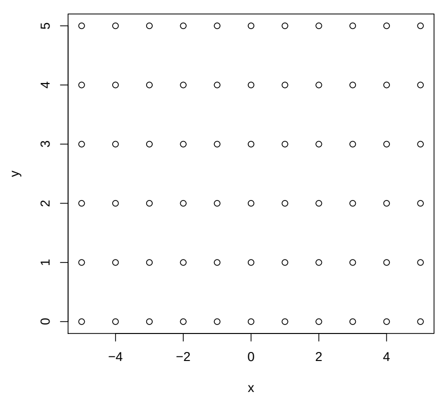
2. Theorem 2

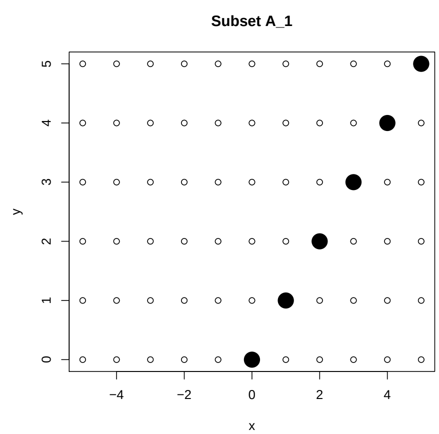
$$
\begin{aligned}
\text{LHS} &= A \cap (A \cup B) \\
&= (A \cup \emptyset) \cap (A \cup B) \;\; \text{From Axiom 4} \\
&= A \cup (B \cap \emptyset) \;\; \text{From Axiom 3 and Axiom 1} \\
&= A \cup \emptyset \text{ From Axiom 4} \\
&= A \\
&= \text{RHS}
\end{aligned}
$$

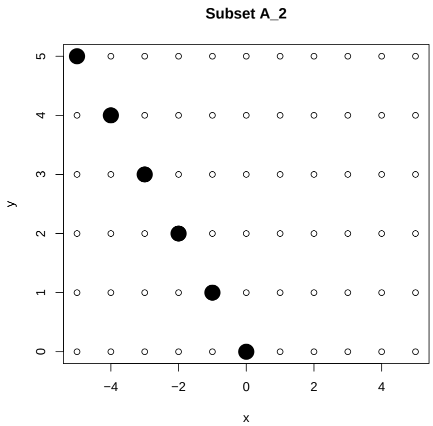
4. Let $S = [(x, y) \in \Re^2; -5 \leq x \leq 5, 0 \leq y \leq 5, x, y = \text{integers}]$.

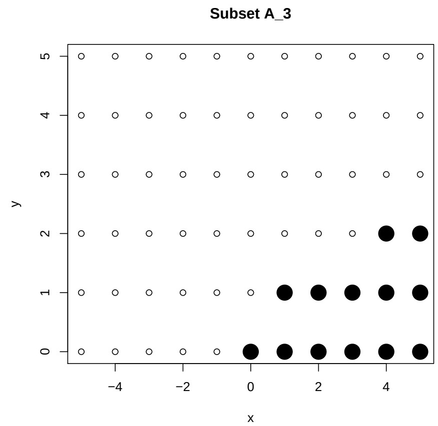
1. List (or illustrate) S
2. List the elements of the set $A_1 = [(x, y) \in S; x = y]$
3. List the elements of the set $A_2 = [(x, y) \in S; x = -y]$
4. List the elements of the set $A_3 = [(x, y) \in S; x \geq y^2]$
5. List the elements of $A_1 \cup A_2$.
6. List the elements of $A_1 \cap A_2$
7. List the elements of $A_1 \cup A_2 \cup A_3$

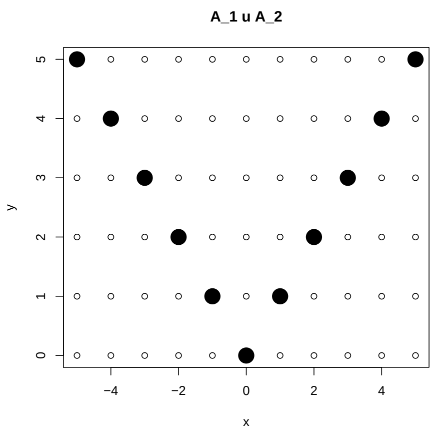
Graphically illustrating each set elements..

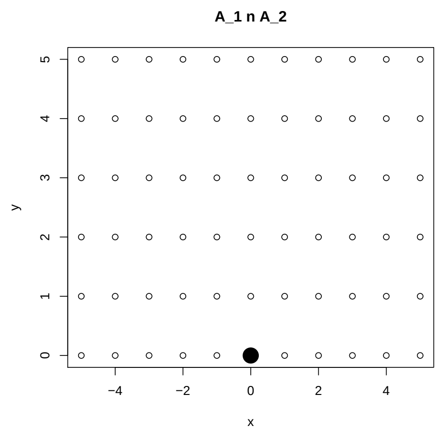


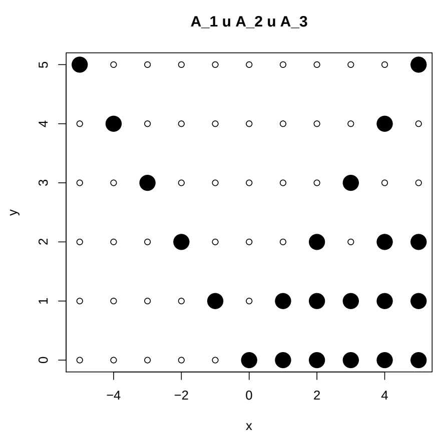












Numerically

1. $\{(-5,0), (-5,1), \ldots, (5,5)\}$
2. $\{(0,0), (1,1), (2,2), (3,3), (4,4), (5,5)\}$
3. $\{(-5,5), (-4,4), (-3,3), (-2,2), (-1,1), (0,0)\}$

4. $\{(0,0),(1,0),(2,0),(3,0),(4,0),(5,0),(1,1),(2,1),(3,1),(4,1),(5,1),(4,2),(5,2)\}$

5. $\{(-5,5),(-4,4),(-3,3),(-2,2),(-1,1),(0,0),(1,1),(2,2),(3,3),(4,4),(5,5)\}$

6. $\{(0,0)\}$

7. $\{(-5,5),(-4,4),(-3,3),(-2,2),(-1,1),(0,0),(1,0),(2,0),(3,0),(4,0),(5,0),$
$(1,1),(2,1),(3,1),(4,1),(5,1),(2,2),(4,2),(5,2),(3,3),(4,4),(5,5)\}$