Math 324 Sept 10, 2004 ①

<u>Homework #2</u> Due <u>sept 24</u>
chapter 2 prob 8, 12, 19, 31, 40, 45, 63, 82, 106

<u>Axioms of Probability</u>

1. for any event A, $P(A) \geq 0$
2. $P(S) = 1$
3. If $A_1, A_2, \ldots, A_k$ is a finite collection of mutually exclusive events, then
$$P(A_1 \cup A_2 \cup \cdots \cup A_k) = \sum_{i=1}^{k} P(A_i)$$

If $A_1, A_2, \ldots$ is an infinite collection of Mutually exclusive events then
$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} P(A_i)$$

<u>Example</u>

Roll a 6 sided dice

$S = \{1, 2, 3, 4, 5, 6\}$

$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$

Probability Theorems

T1 $P(A) = 1 - P(A^c)$

Proof

by axiom 3a

let $A_1 = A$, $A_2 = A^C$

Now from set theory

$A \cup A^c = S$

$1 = P(S) = P(A \cup A^c) = P(A) + P(A^c)$

$\Rightarrow P(A) = 1 - P(A^c)$

T2 If A and B are mutually exclusive, then $P(A \cap B) = 0$

Proof

$A \cap B = \emptyset$

$\Rightarrow (A \cap B)^c = \emptyset^c$

$\Rightarrow (A \cap B)^c = S$

$P(S) = 1$

$\Rightarrow P((A \cap B)^c) = 1$

Theorem 1 $\Rightarrow P(A \cap B) = 1 - P((A \cap B)^c)$

$= 1 - 1$

$= 0$.

T3 For any two events A and B,

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

Proof

$A\cup B = A\cup(B\cap A^c)$ (check using set theory)

A and $(B\cap A^c)$ are mutually exclusive

$\Rightarrow$ From axiom 3a

$$P(A\cup B)=P(A\cup(B\cap A^c)) = P(A)+P(B\cap A^c)$$

Note also that $B = (B\cap A)\cup(B\cap A^c)$ (check using set theory)

$$\Rightarrow P(B) = P((B\cap A)\cup(B\cap A^c))$$
$$= P(B\cap A)+P(B\cap A^c)$$

$$\Rightarrow P(B\cap A^c) = P(\underline{B})\,P(B\cap A)$$

$$\Rightarrow P(A\cup B)=P(A)+(P(B)-P(B\cap A))$$

count counts
count twice.

T4

If $A \subset B$ then $P(A) \leq P(B)$

proof Since $A \subset B$ $\quad B = A \cup (B \cap A')$

~~by the~~ which is disjoint therefore by axiom 3

$$P(B) = P(A \cup (B \cap A)') = P(A) + P(B \cap A')$$

By axiom $P(B \cap A') \geqslant 0$

$$\Rightarrow P(B) \geqslant P(A)$$

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T5 Boole's Inequality

If $A_1, A_2, \ldots$ is a sequence of events

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$$

Proof

Let $B_1 = A_1$, $B_2 = A_2 \cap A_1'$, $B_i = A_i \cap \left(\bigcup_{j=1}^{i-1} A_j\right)^c$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

Also $B_1, B_2, \ldots$ are mutually exclusive

Note that $B_i \subset A_i$ therefore from T4 that $P(B_i) \leq P(A_i)$ therefore

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \underbrace{P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i)}_{\text{axiom 3}} \leq \sum_{i=1}^{\infty} P(A_i)$$

T7 Bonferroni's Inequality

If $A_1, A_2, \ldots A_k$ are events then

$$P\left(\bigcap_{i=1}^{k} A_i\right) \geq 1 - \sum_{i=1}^{k} P(A_i^c)$$

Proof exercise

use the relationship

$$\bigcap_{i=1}^{k} A_i = \left(\bigcup_{i=1}^{k} A_i^c\right)^c$$

apply T1 and T6.

Determining Probabilitys

Suppose we have many simple events E_i such that

$P(E_i) \geq 0$ and $\sum_i P(E_i) = 1$

Let A be a compound event then

$$P(A) = \sum_{E_i \in A} P(E_i)$$

ie the probability of compound event A is the sum of the probabilitys of the simple events $E_i \in A$.

Equally likely outcomes

If there is N possible simple outcomes and each is equally likely with $p = P(E_i)$ then

$$1 = \sum_{i=1}^{N} P(E_i) = \sum p = Np \Rightarrow p = \frac{1}{N}$$

For a compound event A

$$P(A) = \sum_{E_i \in A} P(E_i) = \sum_{E_i \in A} \frac{1}{N} = \frac{N(A)}{N}$$

$N(A) \equiv$ Number of outcomes in A.

Example toss 2 fair coins

$$S = \{HH, HT, TH, TT\}$$

Every outcome has $\frac{1}{4}$ of happening

A = two of same face = $\{HH, TT\}$

P(two of same face) $= \frac{2}{4} = \frac{1}{2}$