Lecture 13

We have been talking about discrete probability distributions. Including discussing the pdf and cdf

ie $p(x) = P(X = x)$

$$F(x) = \sum_{y:y\leq x} p(y)$$

Today we discuss how to compute the mean and variance of a discrete random variable using knowledge about the distribution.

<u>Expected Value</u>

If X is a discrete r.v. with pdf $p(x)$ and possible values $x_1, x_2, \ldots (x_k)$ then the <u>expected value</u> of X is

$$E(X) = \mu_X = \sum_{x_i} x_i \, p(x_i)$$

Another terminology is that $E(X)$ is the <u>mean value</u> of the r.v X

Variance

If X is a discrete r.v. with pdf $p(x)$ and possible values $x_1, x_2, \ldots$ then the variance of X is

$$Var(X) = \sum_{x_i} (x_i - \mu)^2 p(x_i)$$

Alternative symbol σ_X^2

Note that for computation it is often easier to use the formula:

$$Var(X) = E[X^2] - [E(X)]^2$$

where $E[X^2] = \sum_{x_i} x_i^2 p(x_i)$

Expection/variance for functions of random variables

Suppose that X is a discrete r.v with pdf $p(x)$ and possible values $x_1, x_2, \ldots$ ~~that the variance~~ Suppose that $h(X)$ is a function of the random variable.

Then

$$E[h(X)] = \sum_{x_i} h(x_i)\,p(x_i)$$

and

$$Var(h(x)) = \sum_{x_i} \left[h(x_i) - E[h(X)]\right]^2 p(x)$$

Special cases

$$E[aX+b] = aE[X]+b$$

constants

$$Var(aX+b) = a^2 Var(X)$$

Some Examples

1. Roll 2 fair 6 sided dice. X = "sum of two faces"

x	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$E[X] = 2\left(\tfrac{1}{36}\right) + 3\left(\tfrac{2}{36}\right) + 4\left(\tfrac{3}{36}\right) + \cdots + 11\left(\tfrac{2}{36}\right) + 12\left(\tfrac{1}{36}\right)$$

$$= \frac{252}{36} = 7$$

$$Var(X) = E[X^2] - (E(X))^2$$

$$E[X^2] = 2^2\left(\frac{1}{36}\right) + 3^2\left(\frac{2}{36}\right) + 4^2\left(\frac{3}{36}\right) + \cdots + 11^2\left(\frac{2}{36}\right) + 12^2\left(\frac{1}{36}\right)$$

$$= \frac{1974}{36} = 54.833$$

$$Var(X) = 54.833 - 7^2$$

$$= 5.833$$

suppose that we have a new random variable Y which is a function of X

ie $Y = 6X - 3$

What is $E[Y]$, $Var(Y)$?

by the ~~rest~~ rule above

$$E[Y] = E[6X-3] = 6E[X] - 3 = 6(7) - 3 = 39$$

$$Var(Y) = Var(6X-3) = Var(6X) = 6^2 Var(X) = 6^2(5.833) = 210$$

Note that you could also check this using (5)

y	$9 = 6(2)-3$	$15 = 6(3)-3$	21	27	33	39	45	51	57	63	69
$P(Y=y)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

2. Suppose Z is a Bernoulli random variable with $P(Z=1)=p$

so pdf

z	0	1
$p(z)$	$(1-p)$	p

$$E(Z) = 0(1-p) + 1p = p$$

$$Var(Z) = E[Z^2] - (E(Z))^2$$

$$E[Z^2] = 0^2(1-p) + 1^2 p = p$$

$$\Rightarrow Var(Z) = p - p^2 = p(1-p)$$

3. Consider the baseball player from the previous lecture

$$p(x) = \begin{cases} (1-p)^{x-1} p & x = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \sum_{x=1}^{\infty} x(1-p)^{x-1} p$$

$$= p \sum_{x=1}^{\infty} x(1-p)^{x-1}$$

note that $x(1-p)^{x-1} = -\frac{d}{dp}(1-p)^x$

$$= p \sum_{x=1}^{\infty} -\frac{d}{dp}(1-p)^x$$

$$= (p)\left(-\frac{d}{dp} \sum_{x=1}^{\infty} (1-p)^x\right)$$

geometric series

Let $s = a + a^2 + \cdots\cdots$

$$\Rightarrow (1-a)s = a + a^2 + \cdots\cdots - a^2 - a^3 - \cdots$$

$$\Rightarrow s = \frac{a}{(1-a)}$$

$$\Rightarrow \sum_{x=1}^{\infty} (1-p)^x = \frac{1-p}{p}$$

$$\Rightarrow \frac{d}{dp} \sum_{x=1}^{\infty} (1-p)^x = \frac{d}{dp}\left[\frac{1-p}{p}\right] = \frac{(-1)p - (1-p)(1)}{p^2}$$

$$= \frac{-p - 1 + p}{p^2}$$

$$= \frac{-1}{p^2}$$

$$\Rightarrow E[X] = -p\left(-\frac{1}{p^2}\right) = \frac{1}{p}$$

Var(X) is a little too complicated to derive right now

4. Show that

$$E[aX+b] = aE[X]+b$$
$$Var(aX+b) = a^2 Var(X)$$

$$E[aX+b] = \sum_{x_i} (ax_i + b)\, p(x_i)$$
$$= \sum_{x_i} a x_i p(x_i) + \sum_{x_i} b\, p(x_i)$$
$$= a \sum_{x_i} x_i p(x_i) + b \underbrace{\sum_{x_i} p(x_i)}_{=1}$$
$$= aE(X) + b$$

$$Var(aX+b) = \sum_{x_i} \left(ax_i + b - E[aX+b]\right)^2 p(x_i)$$
$$= \sum_{x_i} (ax_i + b - aE(X) + b)^2 p(x_i)$$
$$= \sum_{x_i} (ax_i - aE(X))^2 p(x_i)$$
$$= a^2 \sum_{x_i} (x_i - E(X))^2 p(x_i)$$
$$= a^2 Var(X)$$