

Lecture 13

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We have been talking about discrete probability distributions. Including discussing the pdf and cdf

$$\text{i.e. } p(x) = P(X=x)$$

$$F(x) = \sum_{y: y \leq x} p(y)$$

Today we discuss how to compute the mean and variance of a discrete random variable using knowledge about the distribution.

Expected Value

IF X is a discrete r.v. with pdf $p(x)$ and possible values x_1, x_2, \dots, x_k then the expected value of X is

$$E(X) = M_X = \sum_{x_i} x_i p(x_i)$$

Another terminology is that $E(X)$ is the mean value of the r.v X

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Variance

If X is a discrete r.v. with pdf $p(x)$ and possible values x_1, x_2, \dots then the variance of X is

$$\text{Var}(X) = \sum_{x_i} (x_i - \mu)^2 p(x_i)$$

Alternative symbol σ_X^2

Note that for computation it is often easier to use the formula:

$$\text{Var}(X) = E[X^2] - [E(X)]^2$$

where $E[X^2] = \sum_{x_i} x_i^2 p(x_i)$

Expectation/Variance for functions of random variables

Suppose that X is a discrete r.v. with pdf $p(x)$ and possible values x_1, x_2, \dots

~~that the variance~~ Suppose that $h(X)$ is a function of the random variable.

Then

$$E[h(X)] = \sum_{x_i} h(x_i) p(x_i)$$

and

$$\text{Var}(h(x)) = \sum_{x_i} [h(x_i) - E[h(X)]]^2 p(x)$$

Special cases

$$E[aX + b] = aE[X] + b$$

↑ ↑
constants

$$\text{Var}(aX + b) = a^2 \text{Var}(X)$$

Some Examples

1. Roll 2 fair 6 sided dice. $X =$ "sum of two faces"

x	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\begin{aligned} E[X] &= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + \dots + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right) \\ &= \frac{252}{36} = 7 \end{aligned}$$

$$\text{Var}(X) = E[X^2] - (E(X))^2$$

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$$\begin{aligned} E[X^2] &= 2^2 \left(\frac{1}{36}\right) + 3^2 \left(\frac{2}{36}\right) + 4^2 \left(\frac{3}{36}\right) + \dots + 11^2 \left(\frac{2}{36}\right) + 12^2 \left(\frac{1}{36}\right) \\ &= \frac{1974}{36} = 54.833 \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= 54.833 - 7^2 \\ &= 5.833 \end{aligned}$$

Suppose that we have a new random variable Y which is a function of X

$$\text{i.e. } Y = 6X - 3$$

What is $E[Y]$, $\text{Var}(Y)$?

by the ~~rule~~ rule above

$$\begin{aligned} E[Y] &= E[6X - 3] \\ &= 6E[X] - 3 \\ &= 6(7) - 3 \\ &= 39 \end{aligned}$$

$$\begin{aligned} \text{Var}(Y) &= \text{Var}(6X - 3) \\ &= \text{Var}(6X) \\ &= 6^2 \text{Var}(X) \\ &= 6^2 (5.833) \\ &= 210 \end{aligned}$$

Note that you could also check this using $\binom{6}{2} = 15$ $\binom{6}{3} = 20$...

y	9	15	21	27	33	39	45	51	57	63	69
$P(Y=y)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

2. Suppose Z is a Bernoulli random variable with $P(Z=1) = p$

so pdf

Z	0	1
$P(Z)$	$(1-p)$	p

$$E(Z) = 0(1-p) + 1p = p$$

$$\text{Var}(Z) = E[Z^2] - (E(Z))^2$$

$$E[Z^2] = 0^2(1-p) + 1^2p = p$$

$$\Rightarrow \text{Var}(Z) = p - p^2 = p(1-p)$$

3. Consider the baseball player from the previous lecture

$$p(x) = \begin{cases} (1-p)^{x-1} p & x=1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \sum_{x=1}^{\infty} x(1-p)^{x-1} p$$

$$= p \sum_{x=1}^{\infty} x(1-p)^{x-1}$$

note that $x(1-p)^{x-1} = \frac{d}{dp}(1-p)^x$ (6)

$$= p \sum_{x=1}^{\infty} \frac{d}{dp} (1-p)^x$$

$$= p \left(\frac{d}{dp} \sum_{x=1}^{\infty} (1-p)^x \right)$$

↑
geometric series

$$\text{Let } S = a + a^2 + \dots$$

$$\Rightarrow (1-a)S = a + a^2 + \dots - a^2 - a^3 - \dots$$

$$\Rightarrow S = \frac{a}{(1-a)}$$

$$\Rightarrow \sum_{x=1}^{\infty} (1-p)^x = \frac{1-p}{p}$$

$$\begin{aligned} \Rightarrow \frac{d}{dp} \sum_{x=1}^{\infty} (1-p)^x &= \frac{d}{dp} \left[\frac{1-p}{p} \right] = \frac{(-1)p - (1-p)(1)}{p^2} \\ &= -p - 1 + p \\ &= -\frac{1}{p^2} \end{aligned}$$

$$\Rightarrow E[X] = -p \left(-\frac{1}{p^2} \right) = \frac{1}{p}$$

Var(X) is a little too complicated to derive right now

Q. Show that

$$E[aX+b] = aE[X] + b$$

$$\text{Var}(aX+b) = a^2 \text{Var}(X)$$

$$\begin{aligned} E[aX+b] &= \sum_{x_i} (ax_i + b) p(x_i) \\ &= \sum_{x_i} ax_i p(x_i) + \sum_{x_i} b p(x_i) \\ &= a \sum_{x_i} x_i p(x_i) + b \sum_{x_i} p(x_i) \\ &= aE(X) + b \end{aligned}$$

$$\begin{aligned} \text{Var}(aX+b) &= \sum_{x_i} (ax_i + b - E[aX+b])^2 p(x_i) \\ &= \sum_{x_i} (ax_i + b - aE(X) + b)^2 p(x_i) \\ &= \sum_{x_i} (ax_i - aE(X))^2 p(x_i) \\ &= a^2 \sum_{x_i} (x_i - E(X))^2 p(x_i) \\ &= a^2 \text{Var}(X) \end{aligned}$$

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