Lecture 16

Poisson distribution

A Poisson r.v X has the following pdf

$$p(x;\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad x = 0, 1, 2, \ldots$$

λ is often called the rate parameter
note $\lambda > 0$

Notation

$X \sim Poi(\lambda)$

If $X \sim Poi(\lambda)$ then

$E[X] = \lambda$
$Var(X) = \lambda$

Applications of the Poisson

- typically useful for finding the number of occurences x of an event that occurs over a unit of time, in a unit of volume or space.

  eg - number of phone calls arriving at an exchange in a fixed period of time
  - number of particles emmitted by a radioactive source in a fixed period of time
  - number of customers arriving at a store in an hour

Approximating the binomial

If as $n \to \infty$ and $p \to 0$ $np \to \lambda > 0$

then $b(x; n, p) \to p(x; \lambda)$

ie $$\lim_{n \to \infty} \binom{n}{x} p^x (1-p)^{n-x} = \frac{e^{-\mu} \mu^x}{x!}$$

So we can approximate the binomial distribution using the poisson distribution if n is large and p is small. Rules of thumb are $n \geq 100$, $p \leq 0.01$ $np \leq 20$.

Examples

1. Automobiles arrive at a vehicle emmissions station at a rate of 10 per hour. Suppose that the probability of a vehicle violating emmissions standards is .5.

a) what is the probability that exactly ten arrive during the hour and all ten don't have problems passing the test?

$$P(\text{all ten pass and 10 arrive in a hour})$$

$$= P(\text{all ten pass} \mid \text{10 arrive})\, P(\text{10 arrive})$$

$$= \binom{10}{10}(.5)^{10}(.5)^{0} \frac{e^{-10}\,10^{10}}{10!}$$

$$= (.5^{10})(.125) = .000122$$

b) for any $Y \geq 10$ what is the probability that y arrive in an hour and exactly 10 pass?

$$p(y \text{ arrive and } 10 \text{ pass})$$

$$= P(10 \text{ pass} \mid y \text{ arrive})\, p(y \text{ arrive})$$

$$= \binom{y}{10} .5^{10} (.5)^{y-10} \frac{e^{-\lambda}\lambda^{y}}{y!}$$

$$= \binom{y}{10} .5^{y} \frac{e^{-\lambda}\lambda^{y}}{y!}$$

$$= \frac{y!}{(y-10)!\,10!} \cdot \frac{.5^{y} e^{-\lambda}\lambda^{y}}{y!}$$

$$= \frac{\left(\frac{1}{2}\lambda\right)^{y} e^{-\lambda}}{(y-10)!\,10!}$$

$$= \frac{5^{y} e^{-10}}{(y-10)!\,10!} \quad (\lambda = 10)$$

c) whats the probability that 10 passing cars arrive in the next hour

$$P(10 \text{ passing cars}) = \sum_{y=10}^{\infty} P(y \text{ cars arrive and } 10 \text{ pass})$$

$$= \sum_{y=10}^{\infty} \frac{5^y e^{-10}}{(y-10)!\,10!}$$

$$= \frac{e^{-10}5^{10}}{10!} \sum_{y=10}^{\infty} \frac{5^{y-10}}{(y-10)!}$$

$$= \frac{e^{-10}5^{10}}{10!} \sum_{u=0}^{\infty} \frac{5^u}{u!}$$

(note $e^x = \sum_{u=0}^{\infty} \frac{x^u}{u!}$)

$$= \frac{e^{-10}5^{10}}{10!} e^5$$

$$= \frac{e^{-5}5^{10}}{10!}$$