

# Lecture 16

①

## Poisson distribution

A Poisson r.v.  $X$  has the following pdf

$$p(x; \lambda) = \frac{e^{-\lambda} \lambda^x}{x!} \quad x = 0, 1, 2, \dots$$

$\lambda$  is often called the rate parameter  
note  $\lambda > 0$

Notation

$$X \sim \text{Poi}(\lambda)$$

IF  $X \sim \text{Poi}(\lambda)$  then

$$E[X] = \lambda$$

$$\text{Var}(X) = \lambda$$

## Applications of the Poisson

- typically useful for finding the number of occurrences  $x$  of an event that occurs over a unit of time, in a unit of volume or space.

eg - number of phone calls arriving at an exchange in a fixed period of time

- number of particles emitted by a radioactive source in a fixed period of time

- number of customers arriving at a store in an hour

## Approximating the binomial

If as  $n \rightarrow \infty$  and  $p \rightarrow 0$   $np \rightarrow \lambda > 0$

then  $b(x; n, p) \rightarrow p(x; \lambda)$

$$\text{i.e. } \lim_{n \rightarrow \infty} \binom{n}{x} p^x (1-p)^{n-x} = \frac{e^{-\lambda} \lambda^x}{x!}$$

So we can approximate the binomial distribution using the poisson distribution if  $n$  is large and  $p$  is small. Rules of thumb are  $n \gg 100$ ,  $p \leq 0.01$   $np \leq 20$ .

Examples

1. Automobiles arrive at a vehicle emissions station at a rate of 10 per hour. Suppose that the probability of a vehicle violating emissions standards is .5.

a) what is the probability that exactly ten arrive during the hour and all ten don't have problems passing the test?

$$\begin{aligned}
& P(\text{all ten pass and 10 arrive in a hour}) \\
&= P(\text{all ten pass} | 10 \text{ arrive}) P(10 \text{ arrive}) \\
&= \binom{10}{10} (.5)^{10} (.5)^0 \frac{e^{-10} 10^{10}}{10!} \\
&= (.5^{10}) (.125) = .000122
\end{aligned}$$

b) for any  $y \geq 10$  what is the probability that  $y$  arrive in an hour and exactly 10 pass?

$$P(y \text{ arrive and } 10 \text{ pass})$$

$$= P(10 \text{ pass } | y \text{ arrive}) P(y \text{ arrive})$$

$$= \binom{y}{10} \cdot 5^{10} (0.5)^{y-10} \frac{e^{-\lambda} \lambda^y}{y!}$$

$$= \binom{y}{10} \cdot 5^y \frac{e^{-\lambda} \lambda^y}{y!}$$

$$= \frac{y!}{(y-10)! 10!} \cdot \frac{5^y e^{-\lambda} \lambda^y}{y!}$$

$$= \frac{\left(\frac{\lambda}{2}\right)^y e^{-\lambda}}{(y-10)! 10!}$$

$$= \frac{5^y e^{-10}}{(y-10)! 10!} \quad (\lambda = 10)$$

c) whats the probability that 10 passing cars arrive in the next hour

$$\begin{aligned}
P(10 \text{ passing cars}) &= \sum_{y=10}^{\infty} P(y \text{ cars arrive and } 10 \text{ pass}) \\
&= \sum_{y=10}^{\infty} \frac{5^y e^{-10}}{(y-10)! 10!} \\
&= \frac{e^{-10} 5^{10}}{10!} \sum_{y=10}^{\infty} \frac{5^{y-10}}{(y-10)!} \\
&= \frac{e^{-10} 5^{10}}{10!} \sum_{u=0}^{\infty} \frac{5^u}{u!} \\
&= \frac{e^{-10} 5^{10}}{10!} e^5 \\
&= \frac{e^{-5} 5^{10}}{10!}
\end{aligned}$$

(note  $e^x = \sum_{u=0}^{\infty} \frac{x^u}{u!}$ )