

Lecture 17

①

As mentioned in an earlier lecture there are two types of random variables: discrete and continuous.

We have spent some time studying discrete r.v. so now we shall move onto continuous r.v.

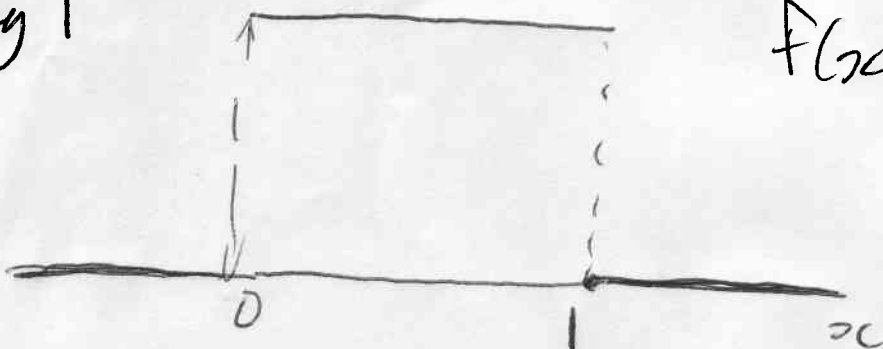
A r.v. X is said to be continuous if its set of possible values is an interval, i.e. for some a, b $a < b$, any value between a and b is possible

A density function is any function where

$$f(x) \geq 0 \quad \forall x$$

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

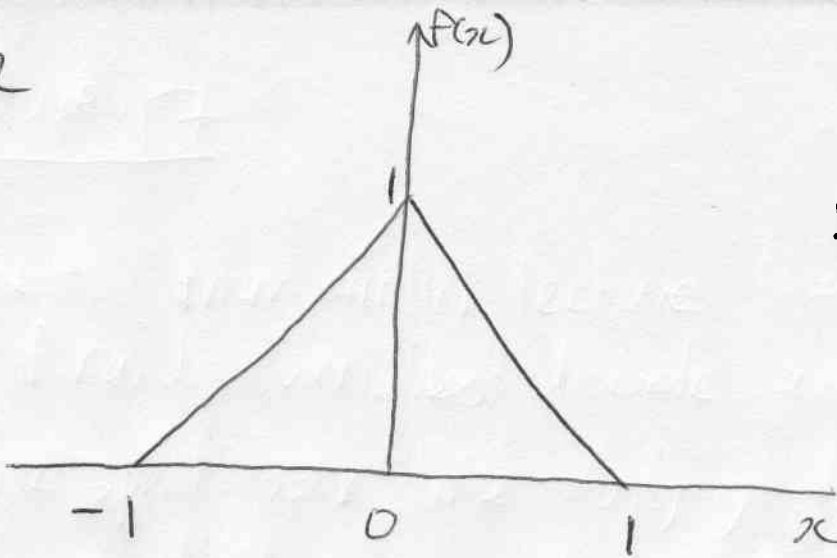
eg 1



$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

eg 2

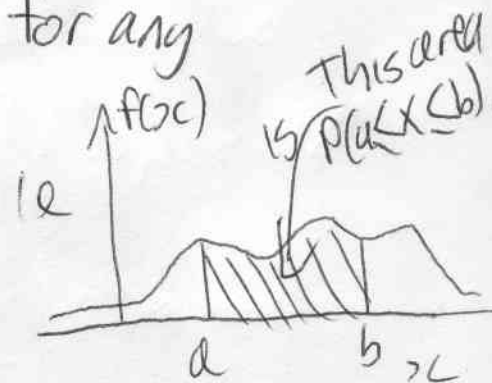
(2)



$$f(x) = \begin{cases} x+1 & -1 \leq x < 0 \\ 1-x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

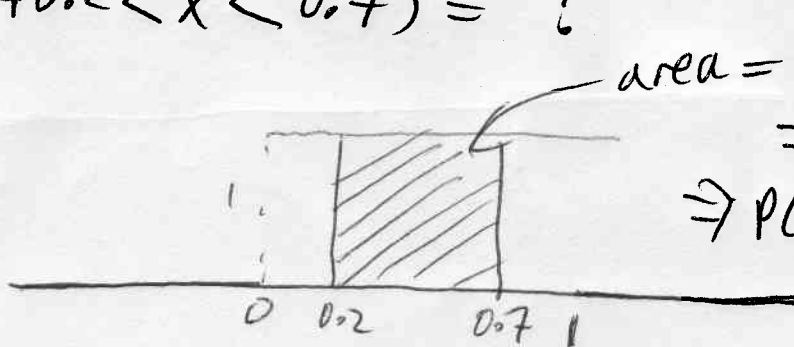
A probability density function (pdf) of a r.v X is the function $f(x)$ such that for any two numbers a and b $a \leq b$

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$



i.e. probabilities are given by areas under the density function

eg for example 1 above $P(+0.2 < X < 0.7) = ?$



$$\begin{aligned} \text{area} &= (0.7 - 0.2)(1) \\ &= 0.5 \end{aligned}$$

$$\Rightarrow P(0.2 < X < 0.7) = 0.5$$

or alternatively

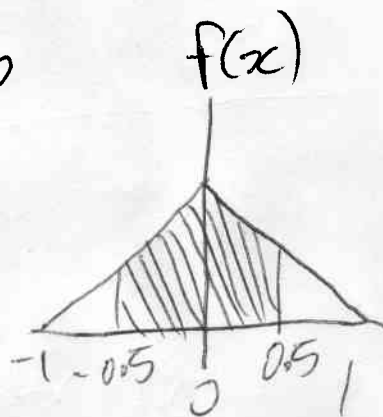
③

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P(0.2 < X < 0.7) = \int_{0.2}^{0.7} 1 \, dx = x \Big|_{0.2}^{0.7} \\ = 0.7 - 0.2 = 0.5$$

eg for example 2

$$f(x) = \begin{cases} x+1 & -1 \leq x < 0 \\ 1-x & 0 \leq x \leq 1 \end{cases}$$



$$P(-0.5 < X < 0.5) = \int_{-0.5}^{0.5} f(x) \, dx \\ = \int_{-0.5}^0 (x+1) \, dx + \int_0^{0.5} (1-x) \, dx \\ = \left[\frac{x^2}{2} + x \right]_{-0.5}^0 + \left[x - \frac{x^2}{2} \right]_0^{0.5} \\ = 0 - (-0.375) + 0.375 \\ = .75$$

Special note

for any number c , and continuous r.v. X

$$P(X=c)=0$$

You can think of this as being true because there is no area under the curve above a single point.

Also note that

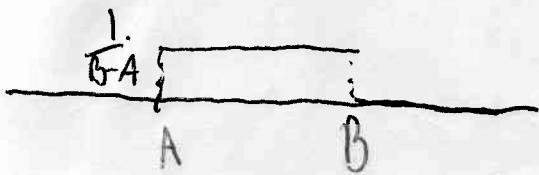
$$P(a < X < b) = P(a \leq X \leq b)$$

are equivalent and equal.

The uniform distribution on $[A, B]$ is defined

by the pdf

$$f(x; A, B) = \begin{cases} \frac{1}{B-A} & A \leq x \leq B \\ 0 & \text{otherwise} \end{cases}$$



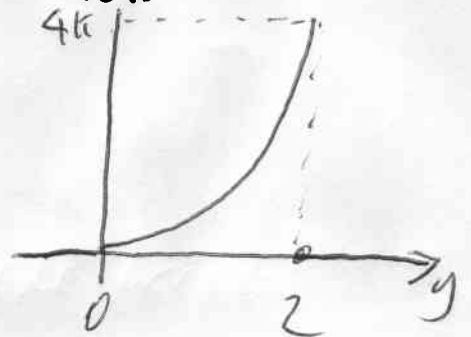
Note eg 1 above is the uniform on $[0, 1]$

Example 1

(5)

Suppose a r.v. Y has pdf $f(y)$

$$f(y) = \begin{cases} ky^2 & 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$



a) Find k

To be valid pdf need $\int_0^2 f(y) dy = 1$

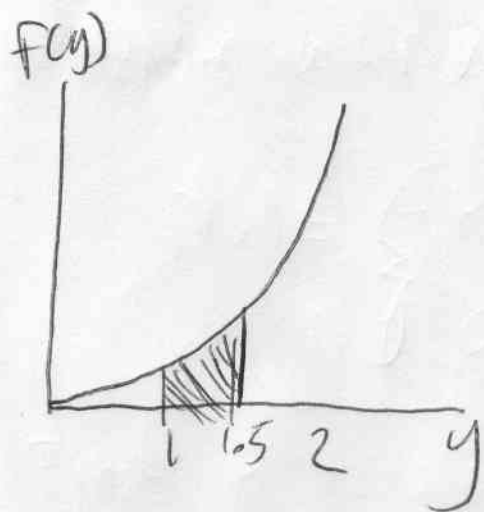
$$\begin{aligned} \int_0^2 ky^2 dy &= k \int_0^2 y^2 dy \\ &= k \left[\frac{y^3}{3} \right]_0^2 \\ &= \frac{8}{3}k \end{aligned}$$

$$\Rightarrow k = \frac{3}{8} \quad (0.375)$$

b) $f(y)$ $P(X \leq 1) = \int_0^1 \frac{3}{8} y^2 dy = \frac{3}{8} \int_0^1 y^2 dy$

$$\begin{aligned} &= \frac{3}{8} \left[\frac{y^3}{3} \right]_0^1 \\ &= \frac{3}{8} \left[\frac{1}{3} - 0 \right] \\ &= \frac{3}{24} \end{aligned}$$

$$c) P(1 \leq X \leq 1.5) = \int_1^{1.5} \frac{3}{8} y^2 dy$$



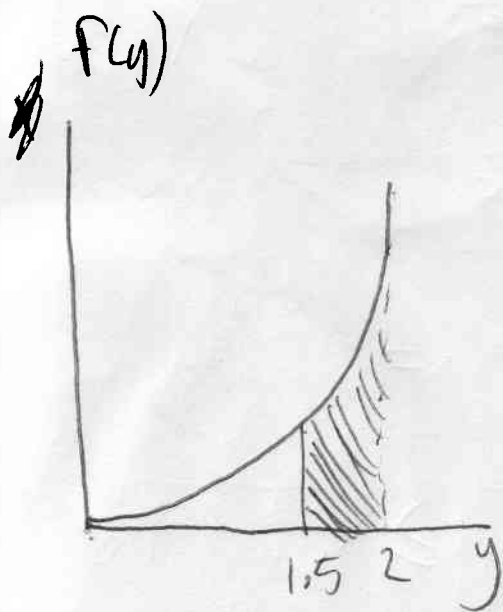
$$= \frac{3}{8} \int_1^{1.5} y^2 dy$$

$$= \frac{3}{8} \left[\frac{y^3}{3} \right]_1^{1.5}$$

$$= \frac{3}{8} \left[\frac{1.5^3}{3} - \frac{1}{3} \right]$$

$$= 0.02969 \text{ (4dp)}$$

$$d) P(X \geq 1.5) = \int_{1.5}^2 \frac{3}{8} y^2 dy$$



$$= \frac{3}{8} \int_{1.5}^2 y^2 dy$$

$$= \frac{3}{8} \left[\frac{y^3}{3} \right]_{1.5}^2$$

$$= \frac{3}{8} \left[\frac{8}{3} - \frac{1.5^3}{3} \right]$$

$$= 0.05791 \text{ (4dp)}$$

Example 2

(7)

Suppose r.v. X has pdf

$$f(x; k, \theta) = \begin{cases} \frac{k\theta^k}{x^{k+1}} & x \geq \theta \\ 0 & x < \theta \end{cases}$$

k, θ are both > 0 and parameters of the pdf for any $\theta < a < b$ Find

$$P(a \leq X \leq b)$$

$$P(a \leq X \leq b) = \int_a^b \frac{k\theta^k}{x^{k+1}} dx$$

$$= k\theta^k \int_a^b \frac{1}{x^{k+1}} dx$$

$$= k\theta^k * \left[-\frac{1}{k} \frac{1}{x^k} \right]_a^b$$

$$= k\theta^k * \left[-\frac{1}{k} \frac{1}{b^k} + \frac{1}{k} \frac{1}{a^k} \right]$$

(8)

$$= \theta^k \left[\frac{1}{a^k} - \frac{1}{b^k} \right]$$

Verify that $f(x)$ is a valid pdf

Let $a \rightarrow 0$ $b \rightarrow \infty$

$$\text{then } \lim_{\substack{a \rightarrow 0 \\ b \rightarrow \infty}} P(a \leq x \leq b) = \int_0^{\infty} \frac{k\theta^k}{x^{k+1}} dx$$

$$= \theta^k \left[\frac{1}{0^k} - \frac{1}{\infty^k} \right]$$

$$= \frac{\theta^k}{\theta^k} = 1$$

So $f(x)$ is valid pdf.