Lecture 17

As mention ~~in~~ an earlier lecture there are two types of random variables: discrete and continuous. We have spent some time studying discrete r.v. so now we shall move onto continuous r.v.

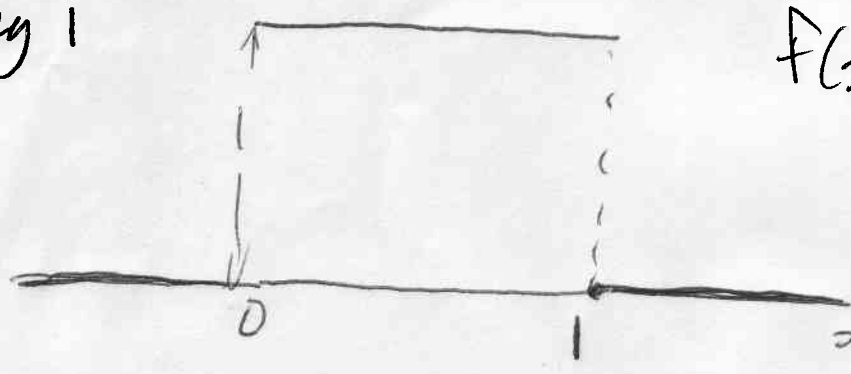
A r.v. X is said to be continuous if its set of possible values is an interval. ie for some a, b $a<b$, any value between a and b is possible

A density function is any function where

$$f(x) \geq 0 \quad \forall x$$

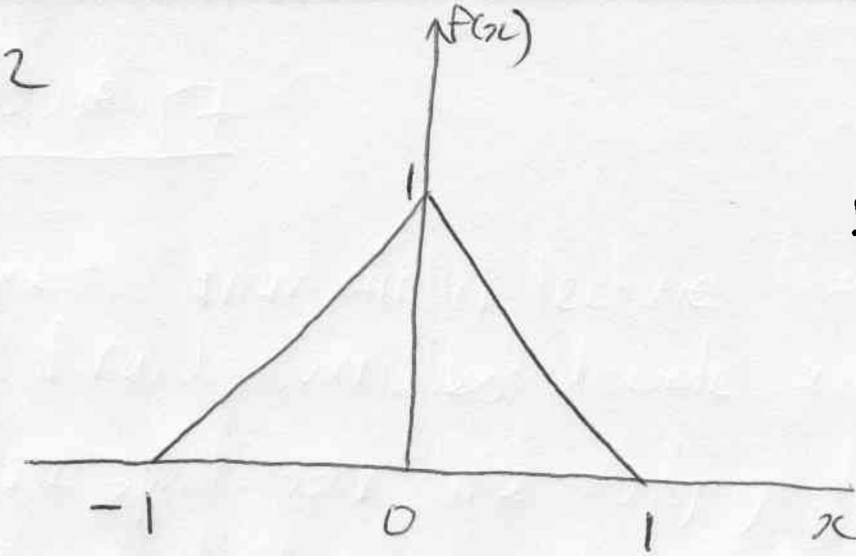
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

eg 1



$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

eg 2

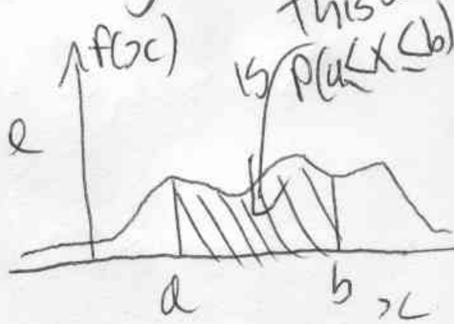


$$f(x) = \begin{cases} x+1 & -1 \leq x < 0 \\ 1-x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

A probability density function (pdf) of a r.v X is the density function $f(x)$ such that for any two numbers a and b $\quad a \leq b$

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

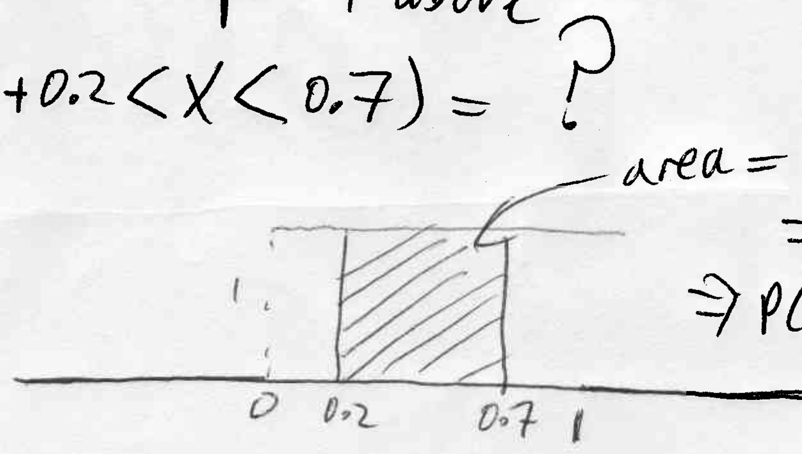
ie



ie probabilities are given by areas under the density function

eg for example 1 above

$P(+0.2 < X < 0.7) = ?$



area $= (0.7-0.2)(1)$

$= 0.5$

$\Rightarrow P(0.2 < X < 0.7)$

$= 0.5$

or alternatively

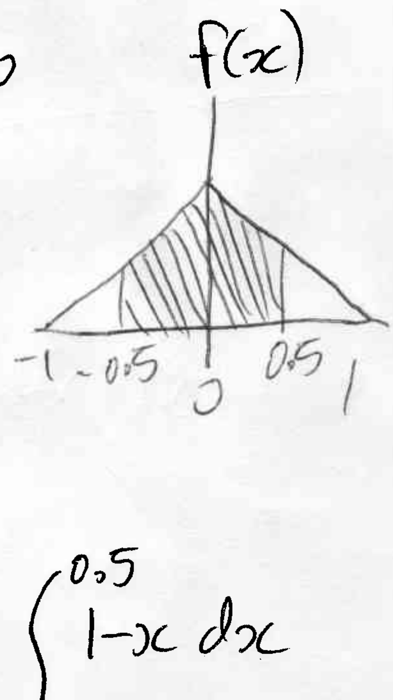
$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P(0.2 < X < 0.7) = \int_{0.2}^{0.7} 1 \, dx = x \Big]_{0.2}^{0.7}$$

$$= 0.7 - 0.2 = 0.5$$

eg for example 2

$$f(x) = \begin{cases} x+1 & -1 \leq x < 0 \\ 1-x & 0 \leq x \leq 1 \end{cases}$$



$$P(-0.5 < X < 0.5) = \int_{-0.5}^{0.5} f(x) \, dx$$

$$= \int_{-0.5}^{0} x+1 \, dx + \int_{0}^{0.5} 1-x \, dx$$

$$= \frac{x^2}{2} + x \Big]_{-0.5}^{0} + x - \frac{x^2}{2} \Big]_{0}^{0.5}$$

$$= 0 - (-0.375) + 0.375$$

$$= .75$$

Special note

for any number c, and continuous r.v. X

$$P(X=c)=0$$

You can think of this as being true because there is no area under the curve above a single point.

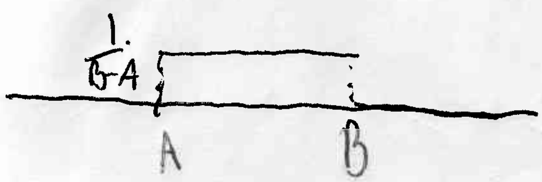
Also note that

$$P(a<X<b)=P(a\leq X\leq b)$$

are equivalent and equal.

The <u>uniform distribution</u> on $[A,B]$ is defined by the pdf

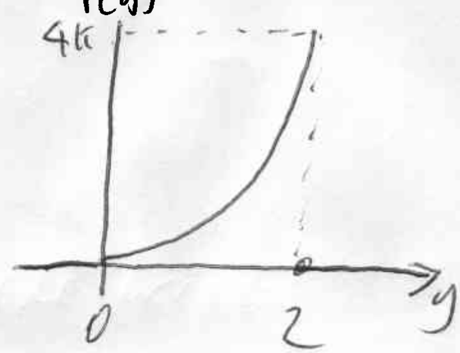
$$f(x;A,B)=\begin{cases}\frac{1}{B-A} & A\leq x\leq B\\ 0 & \text{otherwise}\end{cases}$$



Note eg 1 above is the uniform on $[0,1]$

Example 1

Suppose a r.v Y has pdf f(y)



$$f(y) = \begin{cases} ky^2 & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Find k

To be valid pdf need $\int_0^2 f(x) = 1$

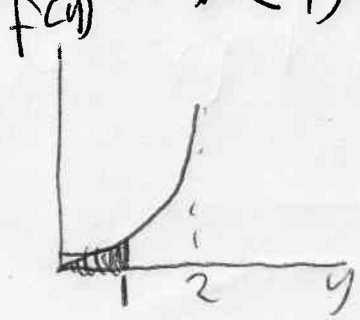
$$\int_0^2 ky^2 = k\int_0^2 y^2\,dy$$

$$= k \left.\frac{y^3}{3}\right]_0^2$$

$$= \frac{8}{3}k$$

$$\Rightarrow k = \frac{3}{8} \quad (.375)$$

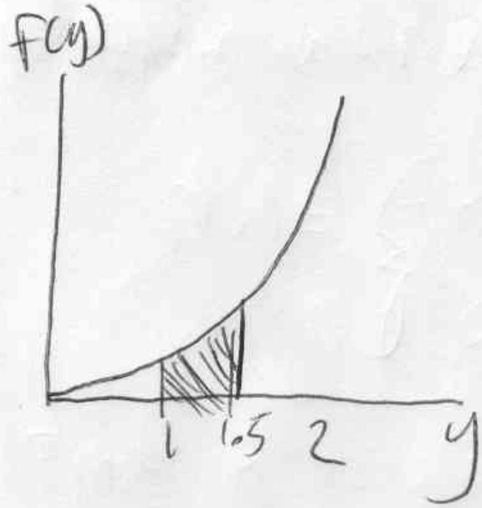
b) $P(Y \le 1) = \int_0^1 \frac{3}{8}y^2\,dy = \frac{3}{8}\int_0^1 y^2\,dy$



$$= \frac{3}{8}\left.\frac{y^3}{3}\right]_0^1 = \frac{3}{8}\left[\frac{1}{3} - 0\right]$$

$$= \frac{3}{24}$$

c) $P(1 \le Y \le 1.5) = \int_{1}^{1.5} \frac{3}{8} y^2 \, dy$



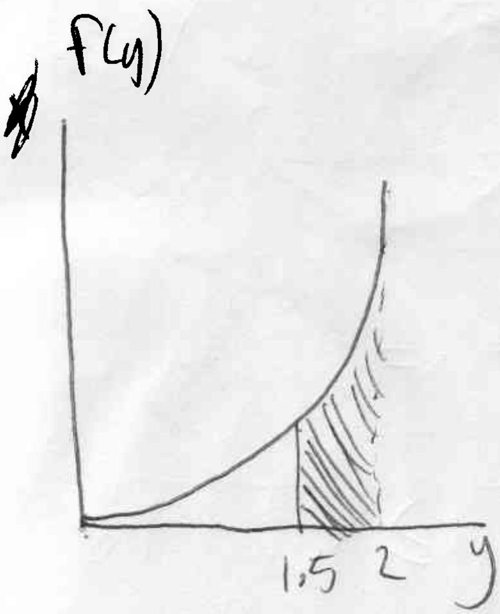
$$= \frac{3}{8} \int_{1}^{1.5} y^2 \, dy$$

$$= \frac{3}{8} \left[\frac{y^3}{3} \right]_{1}^{1.5}$$

$$= \frac{3}{8} \left[\frac{1.5^3}{3} - \frac{1}{3} \right]$$

$$= 0.2969 \quad (4dp)$$

d) $P(Y \ge 1.5) = \int_{1.5}^{2} \frac{3}{8} y^2 \, dy$



$$= \frac{3}{8} \int_{1.5}^{2} y^2 \, dy$$

$$= \frac{3}{8} \left[\frac{y^3}{3} \right]_{1.5}^{2}$$

$$= \frac{3}{8} \quad \frac{8}{3} - \frac{1.5^3}{3}$$

$$= 0.5781 \quad (4dp)$$

Example 2

Suppose r.v X has pdf

$$f(x; k, \theta) = \begin{cases} \dfrac{k\theta^k}{x^{k+1}} & x \geq \theta \\ 0 & x < \theta \end{cases}$$

k, θ are both > 0 and parameters of the pdf for any $\theta < a < b$ Find

$P(a \leq X \leq b)$.

$$P(a \leq x \leq b) = \int_a^b \frac{k\theta^k}{x^{k+1}}\,dx$$

$$= k\theta^k \int_a^b \frac{1}{x^{k+1}}\,dx$$

$$= k\theta^k * -\frac{1}{k}\frac{1}{x^k}\Big]_a^b$$

$$= k\theta^k * \left[-\frac{1}{k}\frac{1}{b^k} + \frac{1}{k}\frac{1}{a^k}\right]$$

$$= \theta^k \left[\frac{1}{a^k} - \frac{1}{b^k}\right]$$

Verify that $f(x)$ is a valid pdf

Let $a \to \theta$ $b \to \infty$

then $\lim\limits_{\substack{a \to \theta \\ b \to \infty}} P(a \leq X \leq b) = \int_{\theta}^{\infty} \frac{k\theta^k}{x^{k+1}}\,dx$

$$= \theta^k \left[\frac{1}{\theta^k} - \frac{1}{\infty^k}\right]$$

$$= \frac{\theta^k}{\theta^k} = 1$$

So $f(x)$ is valid pdf.