

Lecture 18

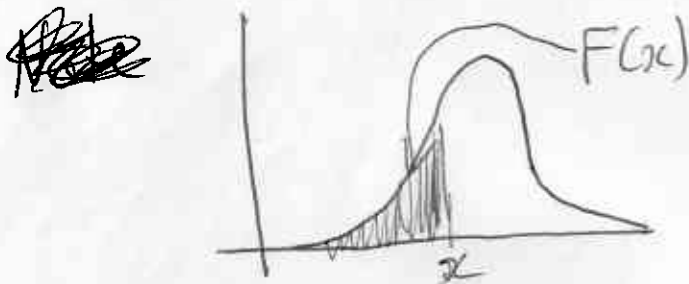
①

The cumulative distribution function cdf $F(x)$ for a continuous r.v. X is defined for every number x by

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(y) dy$$

where $f(y)$ is the density function.

i.e. the CDF $F(x)$ is the area under the density curve to the left of x .



As with discrete r.v

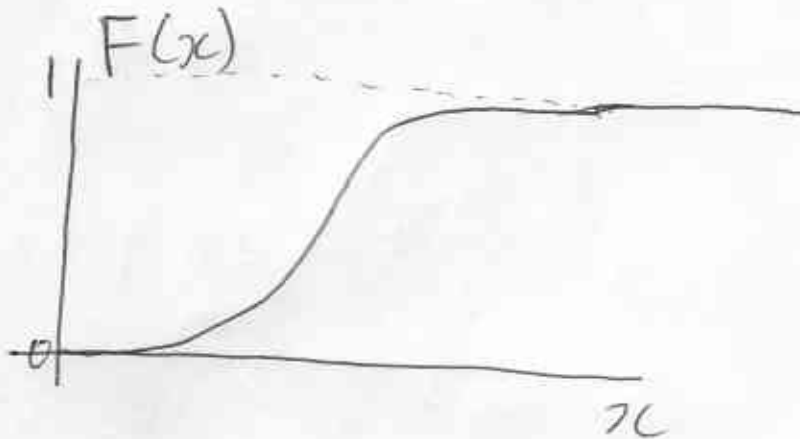
$$F(x) \rightarrow 1 \quad \text{as } x \rightarrow \infty$$

$$F(x) \rightarrow 0 \quad \text{as } x \rightarrow -\infty$$

and if $a < b$

$$F(a) \leq F(b) \quad \text{i.e. } F(x) \text{ is increasing function}$$

Note that graphically $F(x)$ smoothly increases as x increases, unlike the discrete rv case eg



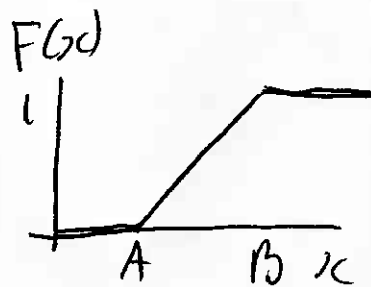
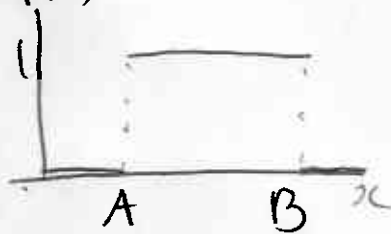
eg Uniform distribution on $[A, B]$. For $A \leq x \leq B$

$$F(x) = \int_{-\infty}^x f(y) dy = \int_A^x \frac{1}{B-A} dy = \frac{1}{B-A} y \Big|_A^x$$

$$= \frac{x-A}{B-A} f(x)$$

$$F(x) = 1 \quad x > B$$

$$F(x) = 0 \quad x < A$$



$$\text{eg } f(x) = \frac{3}{8} x^2 \quad 0 \leq x \leq 2$$

$$0 \quad \text{o.w.}$$

$$F(x) = \dots$$

3

$$F(x) = \int_{-\infty}^x \frac{3}{8} y^2 dy \quad 0 \leq x \leq 2$$

$$= \int_0^x \frac{3}{8} y^2 dy = \frac{3}{8} \frac{y^3}{3} \Big|_0^x$$

$$= \frac{3}{8} \frac{x^3}{3} = \frac{x^3}{8}$$

eg

$$f(x) = \begin{cases} 2(1+x)^{-3} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$F(x) = \int_{-\infty}^x f(y) dy \quad 0 \leq x$$

$$= \int_0^x 2(1+y)^{-3} dy$$

$$= 2 \int_0^x (1+y)^{-3} dy$$

$$= 2 \left[\frac{-1}{2} (1+y)^{-2} \right]_0^x$$

$$= -(1+x)^{-2} - -1$$

$$= 1 - (1+x)^{-2} \quad x \geq 0$$

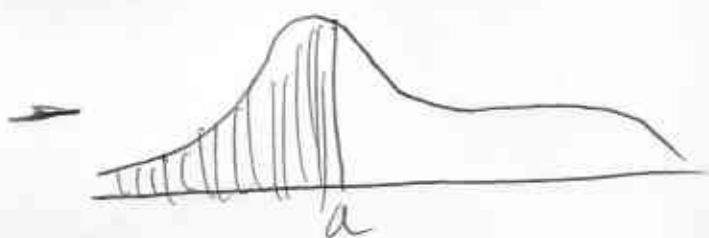
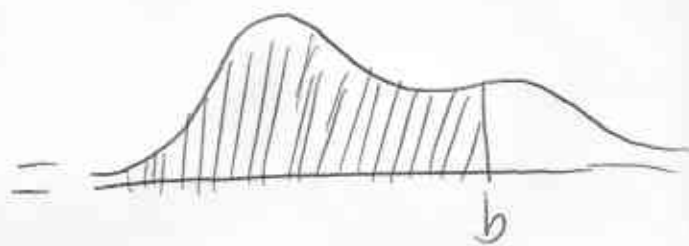
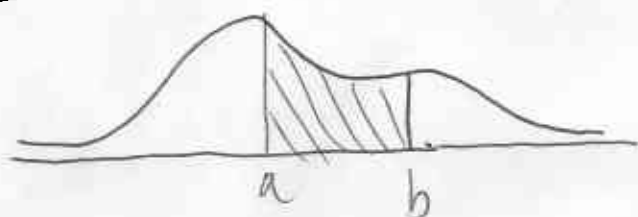
$$F(x) = 0 \quad x < 0$$

Note

$$P(X > a) = 1 - F(a)$$

$$P(a \leq X \leq b) = F(b) - F(a)$$

ie



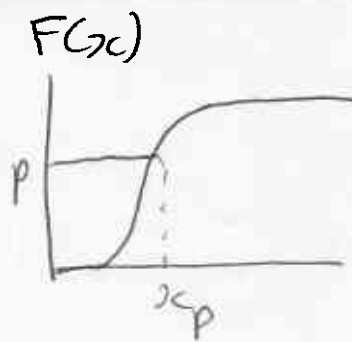
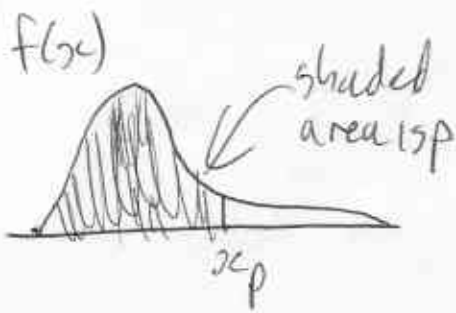
Also $F'(x) = f(x)$ ie differentiate the CDF to get the pdf

Percentiles

100th percentile is the value x_p such that

$$p = \int_{-\infty}^{x_p} f(y) dy$$

$$\text{ie } p = F(x_p)$$



5

eg whats the median of ~~the~~ the distribution

$$F(x) = 1 - (1+x)^{-2} \quad x \geq 0$$

$$0 \quad x < 0$$

$$0.5 = 1 - (1+x_p)^{-2}$$

$$\Rightarrow (1+x_p)^{-2} = 0.5$$

$$\Rightarrow (1+x_p)^2 = 2$$

$$\Rightarrow x_p^2 + 2x_p + 1 = 2$$

$$\Rightarrow x_p^2 + 2x_p - 1 = 0$$

$$\Rightarrow x_p = \frac{-2 \pm \sqrt{4 - 4(1)(-1)}}{2}$$

$$= \frac{-1 \pm \sqrt{2}}{2}$$

only the root $\frac{-1 + \sqrt{2}}{2} = -1 + \sqrt{2}$ is sensible
 $= 0.41$ for median

The expected value of a continuous r.v X with pdf $f(x)$ is

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

eg Uniform on $[A, B]$

$$f(x) = \begin{cases} \frac{1}{B-A} & A \leq x \leq B \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x \frac{1}{B-A} dx \\ &= \int_A^B \frac{x}{B-A} dx \\ &= \frac{x^2}{2(B-A)} \Big|_A^B = \frac{B^2 - A^2}{2(B-A)} \\ &= \frac{(B+A)(\cancel{B-A})}{2(\cancel{B-A})} = \frac{B+A}{2} \end{aligned}$$

eg $f(x) = \begin{cases} 2(1+x)^{-3} & x > 0 \\ 0 & x < 0 \end{cases}$

$$\begin{aligned}
 E[X] &= \int_{-\infty}^{\infty} x(1+x)^{-3} dx \\
 &= \int_0^{\infty} x(1+x)^{-3} dx \quad \text{let } u = 1+x \\
 &= \int_1^{\infty} \frac{u-1}{u^3} du = 2 \int_1^{\infty} \left(\frac{1}{u^2} - \frac{1}{u^3} \right) du \\
 &= 2 \left[-\frac{1}{u} + \frac{1}{2u^2} \right]_1^{\infty} = 2 \left[0 - \left(-1 + \frac{1}{2} \right) \right] = 1
 \end{aligned}$$

The variance of a continuous r.v., X with pdf $f(x)$ is given by

$$\sigma_x^2 = \text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu_x)^2 f(x) dx$$

As before the short cut formula

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

is usually preferred for calculation

(8)

eg $f(x) = \begin{cases} \frac{3}{8}x^2 & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$

$$E[X] = \int_0^2 \frac{3}{8}x^3 dx$$

$$= \frac{3}{8} \frac{x^4}{4} \Big|_0^2 = \frac{3}{8} \cdot 4 = \frac{12}{8} = 1.5$$

$$E[X^2] = \int_0^2 \frac{3}{8}x^4 dx = \frac{3}{8} \frac{x^5}{5} \Big|_0^2$$

$$= \frac{3}{8} \frac{32}{5} = \frac{12}{5} = \frac{24}{10}$$

$$\Rightarrow \text{Var}(X) = E[X^2] - [E(X)]^2$$

$$= \frac{24}{10} - \left(\frac{3}{2}\right)^2$$

$$= \frac{12}{5} - \left[\frac{9}{4}\right]^2$$

$$= \frac{12}{5} - \frac{9}{4} = \frac{48 - 45}{20} = \frac{3}{20}$$