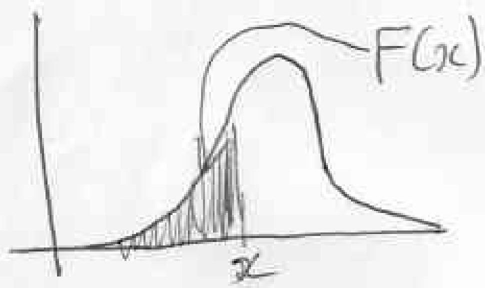
Lecture 18

The cumulative distribution function cdf $F(x)$ for a continuous r.v. X is defined for every number x by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\, dy$$

where $f(y)$ is the density function.

ie the CDF $F(x)$ is the area under the density curve to the left of x.



As with discrete r.v

$F(x) \to 1$ as $x \to \infty$

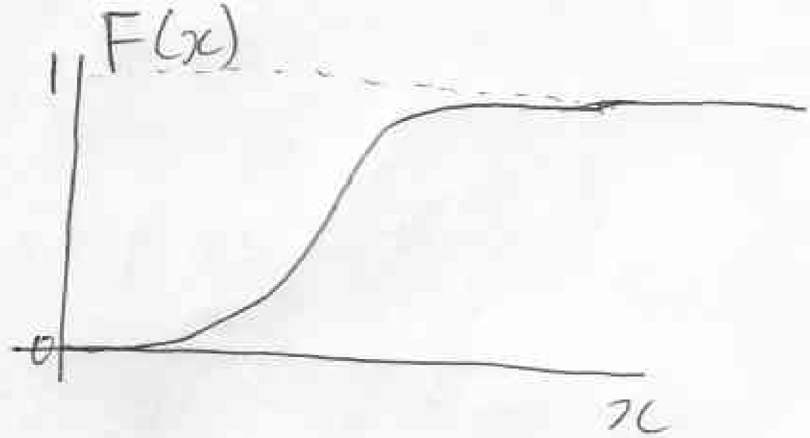
$F(x) \to 0$ as $x \to -\infty$

and if $a < b$

$F(a) \leq F(b)$ ie $F(x)$ is increasing function

Note that graphically $F(x)$ smoothly increases as x increases, unlike the discrete rv case

eg



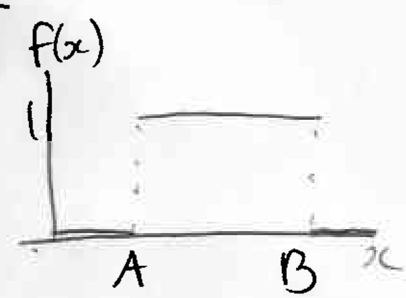
eg Uniform distribution on $[A, B]$. For $A \leq x \leq B$

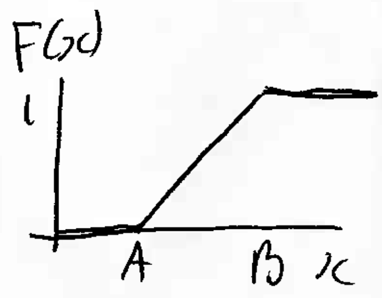
$$F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_{A}^{x} \frac{1}{B-A}\,dy = \frac{1}{B-A}\, y \Big|_{A}^{x}$$

$$= \frac{x-A}{B-A}$$

$F(x) = 1 \quad x > B$

$F(x) = 0 \quad x < A$





eg $f(x) = \frac{3}{8}x^2 \quad 0 \leq x \leq 2$

$\quad\quad\quad\; 0 \quad\quad$ o.w.

$F(x) = ?$

$$F(x) = \int_{-\infty}^{x} \frac{3}{8} y^2 \, dy \qquad 0 \le y \le x \le 2$$

$$= \int_{0}^{x} \frac{3}{8} y^2 \, dy = \frac{3}{8} \frac{y^3}{3} \Big|_{0}^{x}$$

$$= \frac{3}{8} \frac{x^3}{3} = \frac{x^3}{8}$$

eg

$$f(x) = \begin{cases} 2(1+x)^{-3} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(y) \, dy \qquad 0 \le x$$

$$= \int_{0}^{x} 2(1+y)^{-3} \, dy$$

$$= 2 \int_{0}^{x} (1+y)^{-3} \, dy$$

$$= 2 \left[\frac{-1}{2} (1+y)^{-2} \right]_{0}^{x}$$

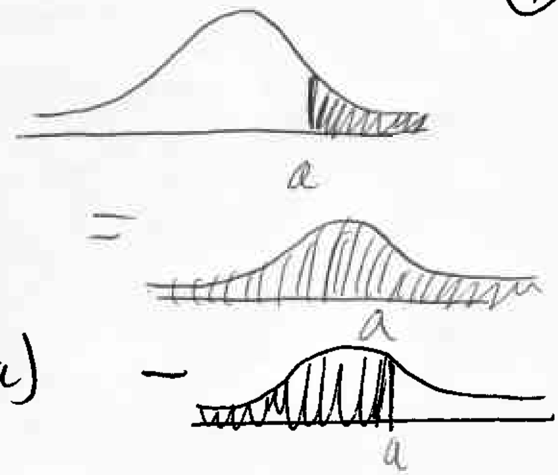
$$= -(1+x)^{-2} - -1$$

$$= 1 - (1+x)^{-2} \qquad x \geq 0$$

$$F(x) = 0 \qquad x < 0$$

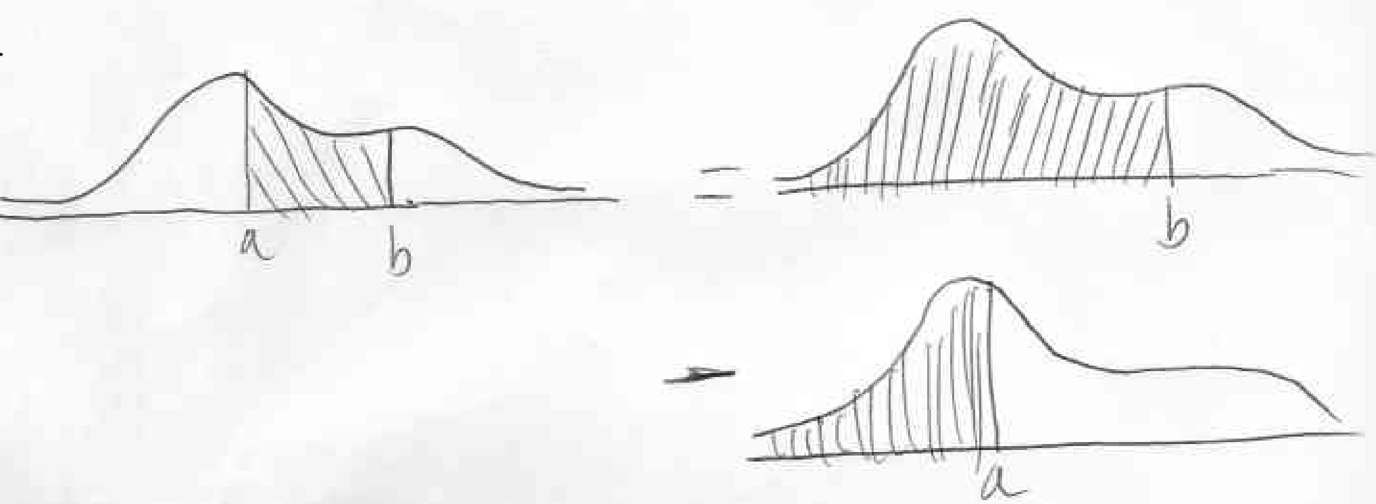
Note

$P(X > a) = 1 - F(a)$



$P(a \leq x \leq b) = F(b) - F(a)$

ie



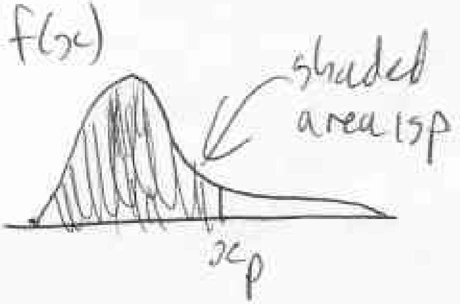
Also $F'(x) = f(x)$ ie differentiate the CDF to get the pdf

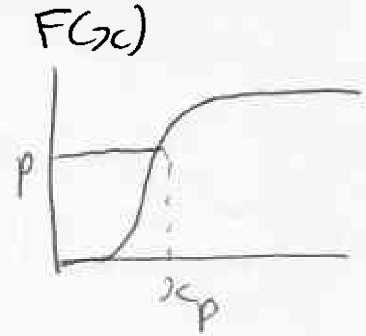
Percentiles

$100p$th percentile is the value x_p such that

$$p = \int_{-\infty}^{x_p} f(y)\,dy$$

ie $p = F(x_p)$





eg whats the median of ~~...~~ the distribution

$$F(x) = 1 - (1+x)^{-2} \qquad x \geq 0$$
$$\qquad\qquad 0 \qquad\qquad x < 0$$

$$0.5 = 1 - (1+x_p)^{-2}$$

$$\Rightarrow (1+x_p)^{-2} = 0.5$$

$$\Rightarrow (1+x_p)^{2} = 2$$

$$\Rightarrow x_p^2 + 2x_p + 1 = 2$$

$$\Rightarrow x_p^2 + 2x_p - 1 = 0$$

$$\Rightarrow x_p = \frac{-2 \pm \sqrt{4 - 4(1)(-1)}}{2}$$

$$= -1 \pm \frac{\sqrt{8}}{2}$$

only the root $-1 + \frac{\sqrt{8}}{2} = -1 + \sqrt{2} = 0.41$ is sensible for median

The <u>expected value</u> of a continuous r.v X with pdf $f(x)$ is

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

eg Uniform on $[A,B]$

$$f(x) = \begin{cases} \frac{1}{B-A} & A \le x \le B \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{B-A}\,dx$$

$$= \int_A^B \frac{x}{B-A}\,dx$$

$$= \left.\frac{x^2}{2(B-A)}\right|_A^B = \frac{B^2-A^2}{2(B-A)}$$

$$= \frac{(B+A)\cancel{(B-A)}}{2\cancel{(B-A)}} = \frac{B+A}{2}$$

eg $$f(x) = \begin{cases} 2(1+x)^{-3} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} 2x(1+x)^{-3}\,dx$$

let $u = 1+x$

$$= \int_{0}^{\infty} 2x(1+x)^{-3}\,dx$$

$$= 2\int_{1}^{\infty} \frac{u-1}{u^3}\,du = 2\int_{1} \frac{1}{u^2} - \frac{1}{u^3}\,du$$

$$= 2\left[-\frac{1}{u} + \frac{1}{2u^2}\right]_1^{\infty} = 2\left[0 - \left(-1 + \frac{1}{2}\right)\right] = 1$$

The variance of a continuous r.v. X with pdf $f(x)$ is given by

$$\sigma_X^2 = Var(X) = \int_{-\infty}^{\infty} (x-\mu_X)^2 f(x)\,dx$$

As before the short cut formula

$$Var(X) = E(X^2) - [E(X)]^2$$ is

usually preferred for calculation

eg $f(x) = \begin{cases} \frac{3}{8}x^2 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$

$$E[X] = \int_0^2 \frac{3}{8}x^3\,dx$$

$$= \frac{3}{8}\left.\frac{x^4}{4}\right|_0^2 = \frac{3}{8}4 = \frac{12}{8} = 1.5$$

$$E[X^2] = \int_0^2 \frac{3}{8}x^4\,dx = \frac{3}{8}\left.\frac{x^5}{5}\right|_0^2$$

$$= \frac{3}{8}\frac{32}{5} = \frac{12}{5}$$

$$\Rightarrow Var(X) = E[X^2] - [E(X)]^2$$

$$= \frac{12}{5} - \left[\frac{3}{2}\right]^2$$

$$= \frac{12}{5} - \frac{9}{4} = \frac{48-45}{20} = \frac{3}{20}$$