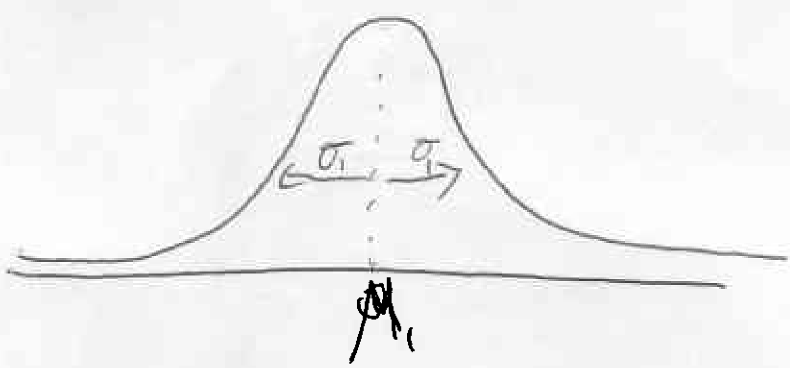
Lecture 19

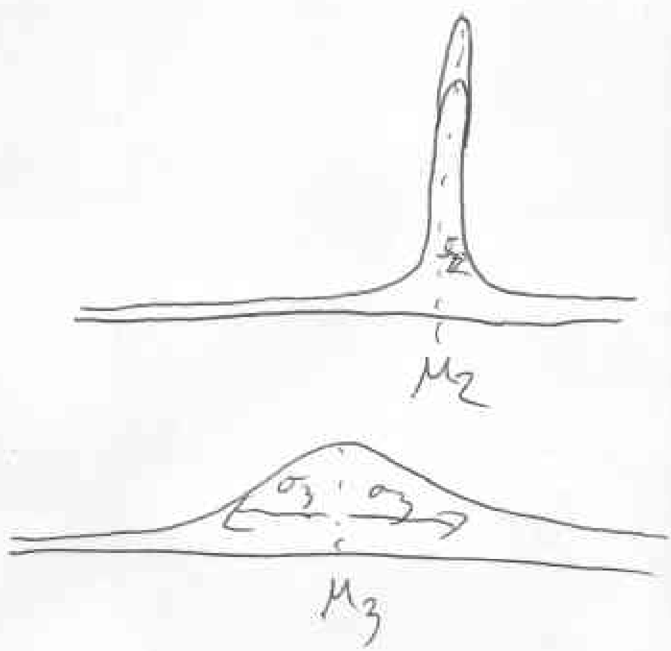
Normal Distribution

A continuous r.v. X has normal distribution with mean μ, standard deviation σ where $-\infty < \mu < \infty$ and $0 < \sigma$ if its pdf is

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad \left(\begin{array}{l}\text{sometimes written}\\ \text{as } \phi(x)\end{array}\right)$$

graphs of the ~~pdf~~ pdf





μ controls location

σ controls spread.

"Symmetric bell shaped" Note μ is also the median for normal distribution

Notation

$X \sim N(\mu, \sigma)$ (sometimes $X \sim N(\mu, \sigma^2)$)

Of course If $X \sim N(\mu, \sigma)$

$$E[X] = \mu$$
$$Var(X) = \sigma^2$$
$$sd(X) = \sigma$$

Applications of the normal

- many physical measurements, heights, weights, ...
- errors in measurements in scientific experiments
- The CLT (to be discussed later) gives that $\bar{x} \sim N(\mu, \sigma)$ as $n \to \infty$
- Normal distribution perhaps most useful in all of statistics

Special Case

If $\mu = 0$ and $\sigma = 1$ then the distribution is called the standard normal distribution.

We usually represent a std normal r.v. using Z, so

$$f(z) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-z^2/2} \qquad -\infty < z < \infty$$

Note that the CDF of Z is

$$P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-y^2/2}\, dy$$

is denoted $\Phi(z)$ ie

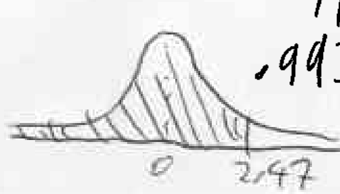
no closed formula for this integral

instead we can use the table (see handout)

eg look up $\Phi(-1.32)$, $\Phi(2.47)$, $\Phi(0.86)$

$\Phi(-1.32) = .0934$

$\Phi(2.47) = .9932$

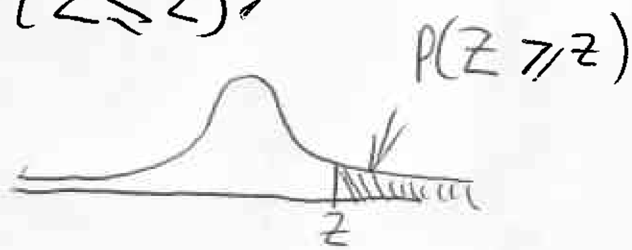


$\Phi(0.86) = .8051$

Since

$\Phi(z) = P(Z \leq z)$ it is easy to look up directly probabilities of the form $P(Z \leq z)$.

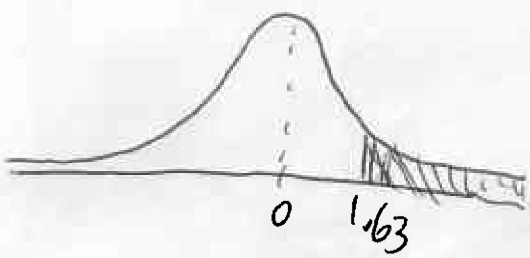
$P(Z \geq z) = ?$ ie

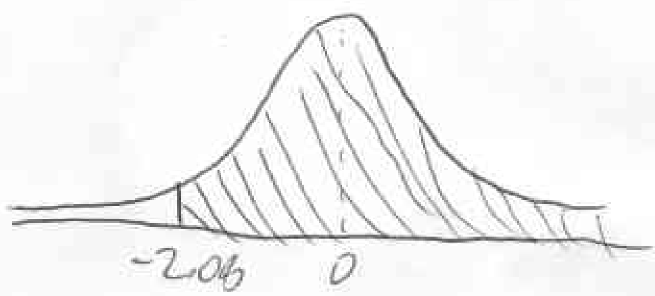


recall from last time

$$P(Z \geq z) = 1 - F(z) = 1 - \Phi(z)$$

what is $P(Z \geq 1.63)$, $P(Z \geq -2.08)$

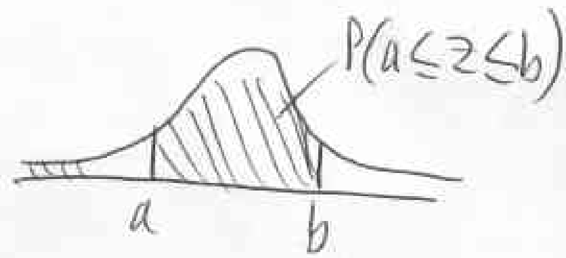




$= 1 - .9484 = .0516$

$= 1 - .0188 = .9812$

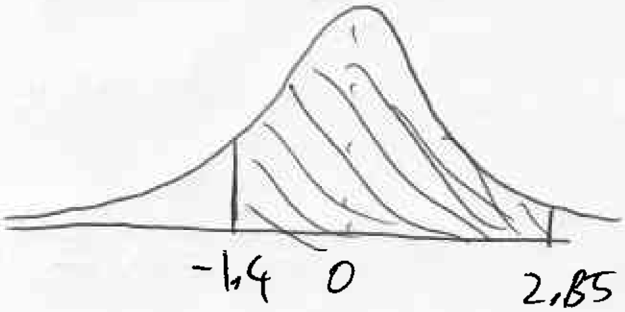
what about $P(a \leq Z \leq b)$?
$a < b$



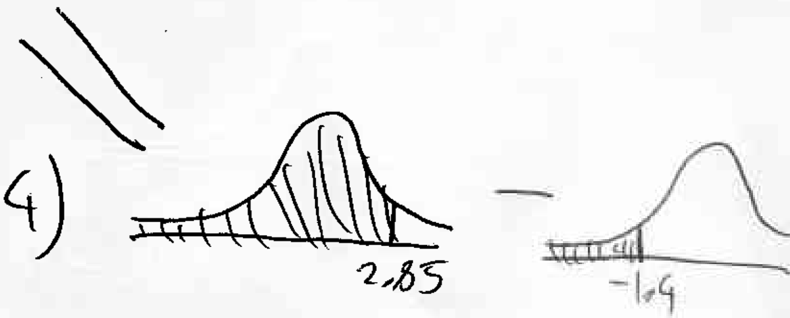
Recall from last time

$$P(a \leq Z \leq b) = F(b) - F(a)$$
$$= \Phi(b) - \Phi(a)$$

What is $P(-1.4 \leq Z \leq 2.85)$

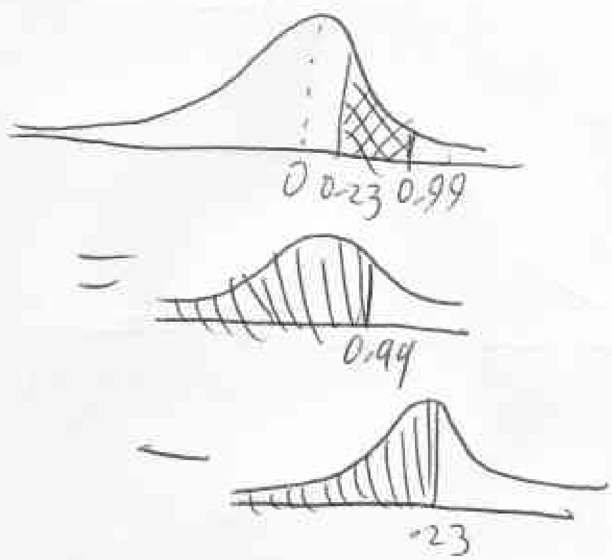


$$= \Phi(2.85) - \Phi(-1.4)$$



$$= .9978 - .0808$$

$$= .9170$$



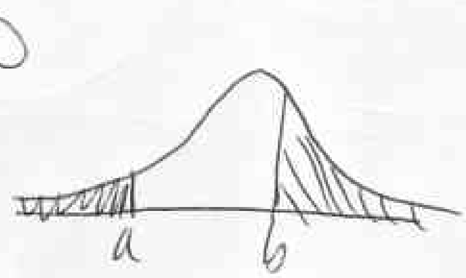
$P(0.23 \leq Z \leq 0.99)$

$$= \Phi(.99) - \Phi(.23)$$

$$= .8389 - .5910$$

$$= .2479$$

What about $P(Z \leq a \text{ or } Z \geq b)$?



$$= P(Z \leq a) + P(Z \geq b)$$

$$= \Phi(a) + 1 - \Phi(b).$$

what if $\mu \neq 0$ and $\sigma \neq 1$?

If X is a r.v with normal distribution mean μ sd σ then

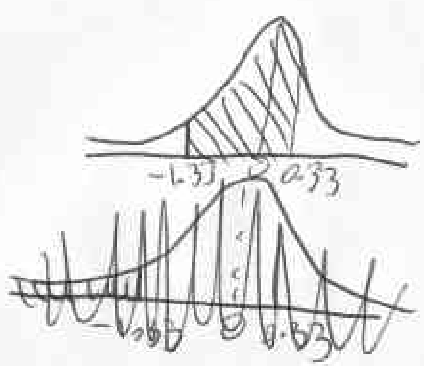
$$Z = \frac{X-\mu}{\sigma}$$ is a r.v with std normal distribution

ie we transform our problem so that we deal with a std normal r.V. and then look up probability in table

eg Suppose $X \sim N(100, 15)$

what is $P(80 \leq X \leq 105)$?

$$P\left(\frac{80-100}{15} \leq \frac{X-100}{15} \leq \frac{105-100}{15}\right)$$



$$= P(-1.33 \leq Z \leq 0.33)$$

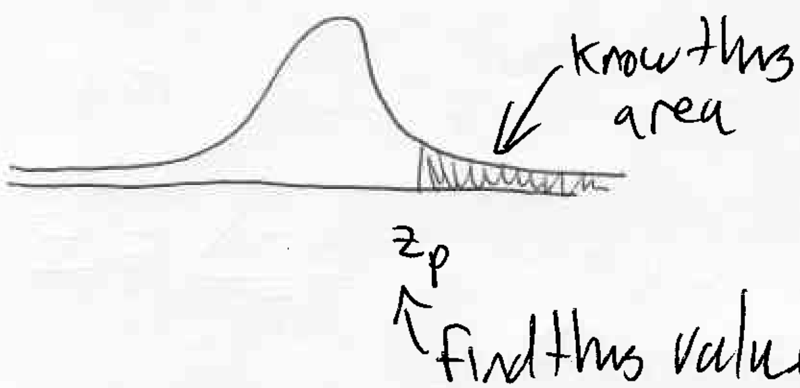
$$= \Phi(0.33) - \Phi(-1.33)$$

$$= .6293 \quad - .0918$$

$$= .5375$$

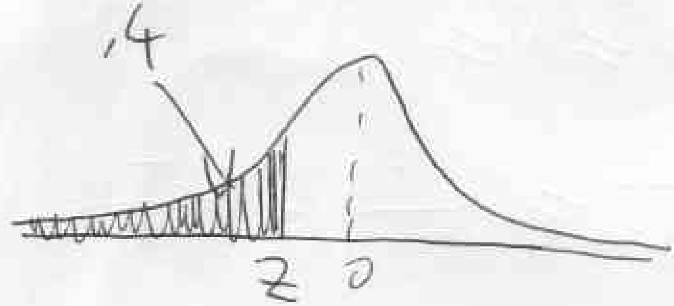
How about the inverse problem?

ie



Easy. Just use the table in the reverse order ie look for the closest probability then find z

eg whats z if $P(Z \leq z) = 0.4$



Closest values are $\Phi(-0.25) = .4013$
and $\Phi(-0.26) = .3974$

so either use $z = -0.25$ or for more accuracy linearly interpolate.

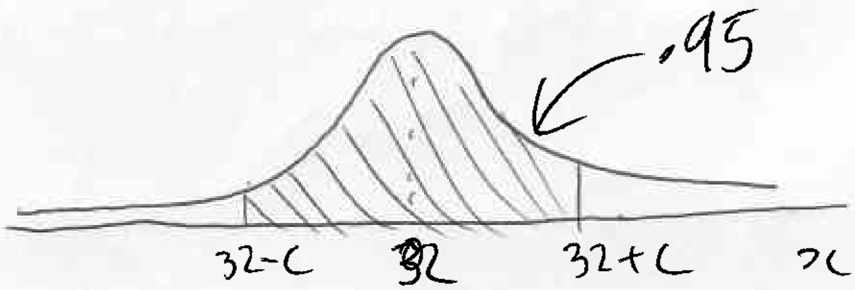
with non standard normal need to use

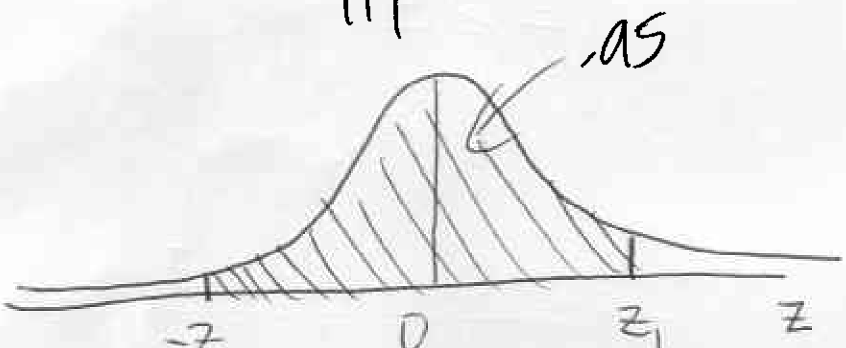
$X = \sigma Z + \mu$ to unstandardize.

eg Suppose $X \sim N(32, 4)$

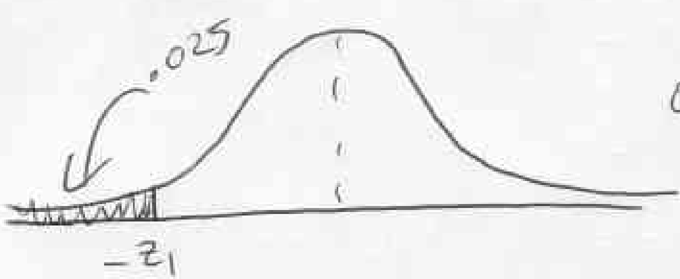
what value of c is

$$P(32 - c \leq X \leq c + 32) = .95$$

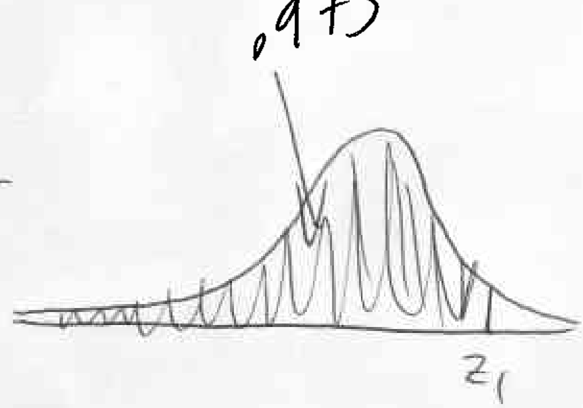




so either



or



from the table

$\Phi(-1.96) = .0250$

$\Phi(1.96) = .9750$

$x_1 = 4(-1.96) + 32 = 24.16$

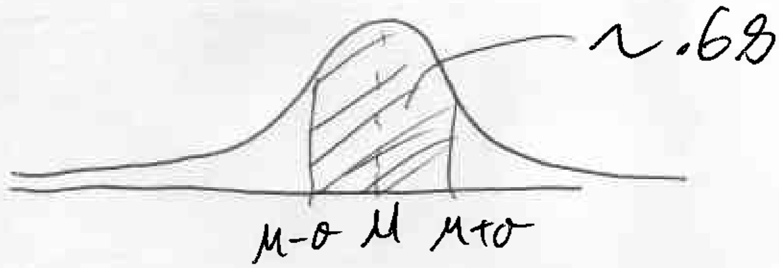
$x_2 = 4(1.96) + 32 = 39.84$

ie $P(24.16 \le X \le 39.84) = .95$

note $c = 4(1.96) = 7.84$

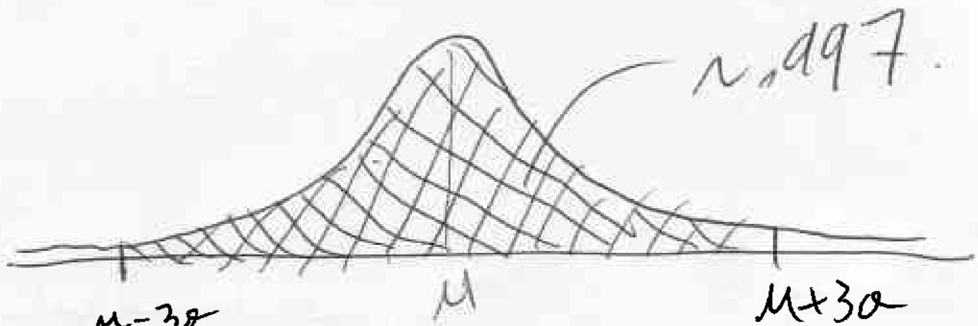
Some empirical rules about normal dist

1 Approximately 68% is within 1SD of the mean



2. Approximately 95% is within 2SD of the mean

3. Approximately 99.7% is within 3SD of the mean



Normal approximation to the binomial

If X is a binomial rv n, p and not too skewed then X has approximately normal distribution $\mu = np$ $\sigma = \sqrt{np(1-p)}$

In practice approximation reasonable if $np \geq 10$ and $n(1-p) \geq 10$.