Lecture 20

An example of approximating binomial with normal

Suppose $X \sim Bin(200, 0.4)$

and we want $P(73 \leq X < 91)$

The normal way $\binom{200}{73} .4^{73} .6^{200-73} + \cdots + \binom{200}{90} .4^{90} .6^{200-90}$

A lot of work. Instead approximate

$$E[X] = 200(0.4) = 80$$

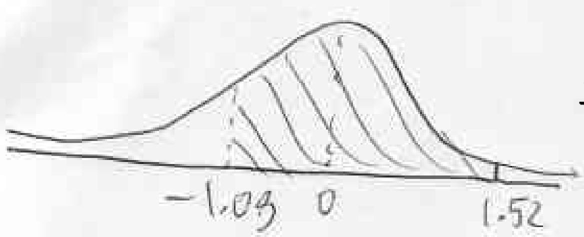
$$SD(X) = \sqrt{200(0.4)(0.6)} = \sqrt{48} = 6.9282$$

continuity correction

$$P(73 \leq X < 91) \approx P(72.5 \leq X \leq 90.5)$$

$$\approx P\left(\frac{72.5-80}{6.9282} \leq Z \leq \frac{90.5-80}{6.9282}\right)$$

$$= P(-1.08 \leq Z \leq 1.52)$$



$$= \Phi(1.52) - \Phi(-1.08)$$

$$= .9357 - .1401$$

$$= .7956$$

The gamma function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$$

where $\alpha > 0$

Note 1. for n an integer $\Gamma(n) = (n-1)!$

2. $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$
3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

A continuous rv X has the <u>gamma distribution</u> if its pdf is

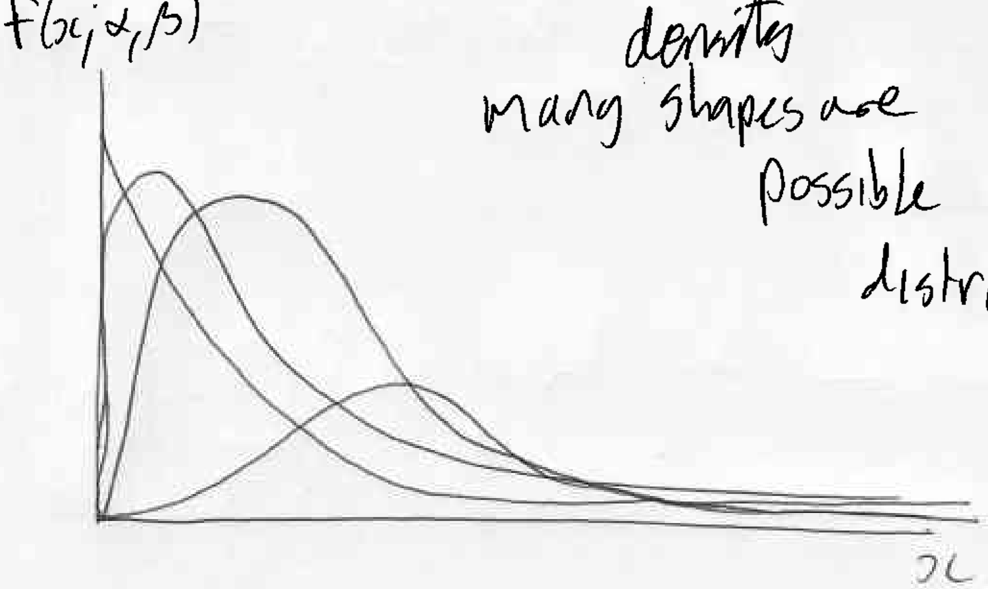
$$f(x;\alpha,\beta) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

with $\alpha > 0$, $\beta > 0$.

shape parameter
scale parameter

<u>Notation</u> $X \sim \text{Gam}(\alpha, \beta)$

2a



density
many shapes are possible with gamma distribution

If $X \sim Gam(\alpha, \beta)$

$E[X] = \alpha\beta$

$Var(X) = \alpha\beta^2$

The function $F(x;\alpha) = \int_0^x \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}\,dy \quad x>0$

is called the incomplete gamma function.

The CDF of the Gamma distribution is

$$P(X \le x) = \int_0^{x/\beta} \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}\,dy$$

Special cases of the gamma distribution

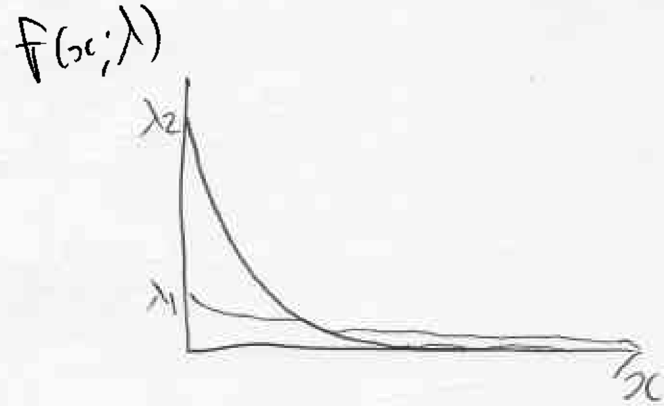
$\alpha = 1 \quad \beta = \frac{1}{\lambda}$ is called the exponential distribution

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

Notation $X \sim Exp(\lambda)$

If $X \sim Exp(\lambda)$ $\quad E[X] = \frac{1}{\lambda} \quad Var(X) = \frac{1}{\lambda^2}$

note some books use $f(x;\lambda) = \frac{1}{\lambda}e^{-x/\lambda}$.



Note for the exponential the CDF is

$$F(x;\lambda) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

(Check by integrating the pdf)

Note It can be shown that if the number of events occuring in a time interval of length t has poisson distribution with parameter λt and the number of occurrences in non overlapping intervals are independent then the distribution of times between arrivals is exponential with parameter $\lambda = \lambda$

$\alpha = \frac{\nu}{2}$ $\beta = 2$ is called the "chi-squared" χ^2 distribution

$$f(x;\nu) = \begin{cases} \dfrac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{(\nu/2)-1} e^{-x/2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Note that ν is called the number of degrees of freedom (df) of X

Note it can be shown that χ^2 distribution with ν df is equivalent to the sum of ν standard normal r.v squared.

Examples

Suppose the time spent by a student on a computer per day is randomly distributed using the gamma distribution with mean 20 min, Variance 80 min^2

a) what is α and β?

X ≡ "time spent on computer"

$E[X] = \alpha\beta = 20$

$Var[X] = \alpha\beta^2 = 80 \Rightarrow \beta = 20/\alpha$

$\Rightarrow \dfrac{400}{\alpha} = 80$

$\Rightarrow \alpha = 5, \beta = 4$

b) What is the probability that the student uses at most 24 minutes on the computer?

$$P(X \leq 24) = \int_0^{24/4} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y}\, dy$$

$$= .715$$ (based on table A.4 in the text)

c) What is the probability that the student uses between 20 and 40 minutes on computer?

$$P(20 \leq X \leq 40) = \int_0^{10} \frac{1}{\Gamma(5)} y^{5-1} e^{-y}\, dy - \int_0^{5} \frac{1}{\Gamma(5)} y^{5-1} e^{-y}\, dy$$

$$= .971 - .560$$

$$= .411.$$