

Lecture 20

①

An example of approximating binomial with normal

Suppose $X \sim \text{Bin}(200, 0.4)$

and we want $P(73 \leq X < 91)$

The normal way

$$\binom{200}{73} \cdot 0.4^{73} \cdot 0.6^{200-73} + \dots + \binom{200}{90} \cdot 0.4^{90} \cdot 0.6^{200-90}$$

A lot of work. Instead approximate

$$E[X] = 200(0.4) = 80$$

$$\text{SD}(X) = \sqrt{200(0.4)(0.6)} = \sqrt{48} = 6.9282$$

$$P(73 \leq X < 91) \stackrel{\text{continuity correction}}{\approx} P(72.5 \leq X \leq 90.5)$$

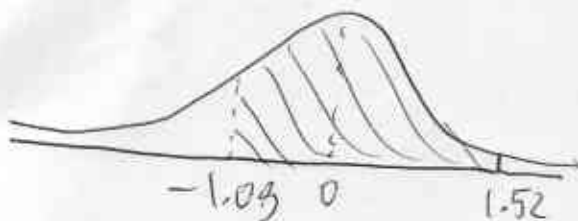
$$\approx P\left(\frac{72.5 - 80}{6.9282} \leq Z \leq \frac{90.5 - 80}{6.9282}\right)$$

$$= P(-1.08 \leq Z \leq 1.52)$$

$$= \Phi(1.52) - \Phi(-1.08)$$

$$= .9357 - .1401$$

$$= .7956$$



The gamma function $\Gamma(x)$ is defined by

$$\Gamma(x) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$$

where $\alpha > 0$

Note 1. for n an integer $\Gamma(n) = (n-1)!$

2. $\Gamma(x) = (x-1)\Gamma(x-1)$

3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

A continuous rv X has the gamma distribution

if its pdf is

$$f(x; \alpha, \beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x > 0 \\ 0 & \text{o.w.} \end{cases}$$

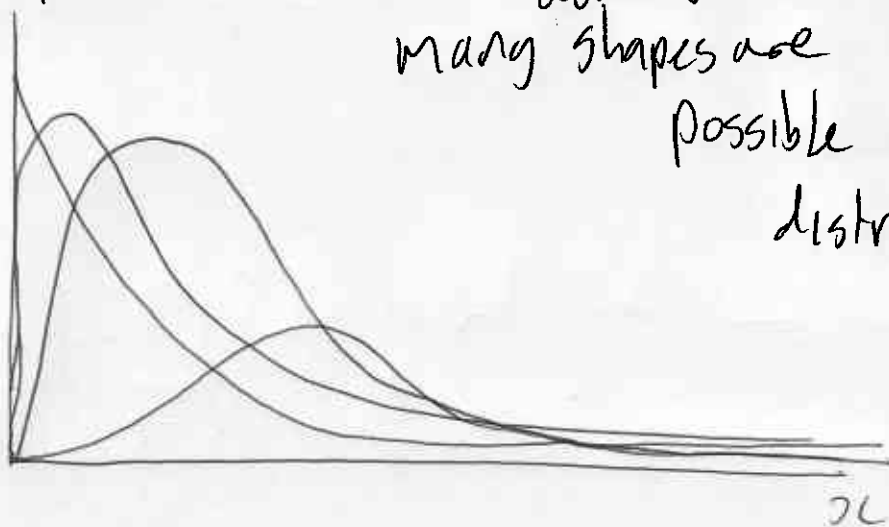
with $\alpha > 0, \beta > 0$.

shape parameter
scale parameter

Notation $X \sim \text{Gam}(\alpha, \beta)$

(2a)

$f(x; \alpha, \beta)$



densities
many shapes are

possible with gamma
distribution

IF $X \sim \text{Gam}(\alpha, \beta)$

$$E[X] = \alpha\beta$$

$$\text{Var}(X) = \alpha\beta^2$$

(3)

The function $\int_0^x \frac{y^{\alpha-1} e^{-y}}{\Gamma(\alpha)} dy$ $x > 0$

is called the incomplete gamma function.

The CDF of the gamma distribution is

$$P(X \leq x) = \int_0^{x/\beta} \frac{y^{\alpha-1} e^{-y}}{\Gamma(\alpha)} dy$$

Special cases of the gamma distribution

$\alpha = 1$ $\beta = \frac{1}{\lambda}$ is called the exponential distribution

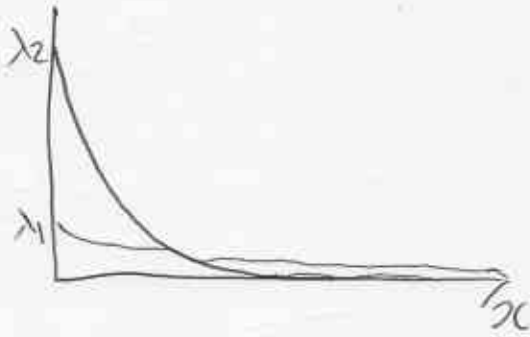
$$f(x; \lambda) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{o.w.} \end{cases}$$

Notation $X \sim \text{Exp}(\lambda)$

$$\text{IF } X \sim \text{Exp}(\lambda) \quad E[X] = \frac{1}{\lambda} \quad \text{Var}(X) = \frac{1}{\lambda^2}$$

note some books use $f(x; \lambda) = \frac{1}{x} e^{-x/\lambda}$.

$f(x; \lambda)$



Note for the exponential the CDF is

$$F(x; \lambda) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

(check by integrating the pdf)

Note It can be shown that if the number of events occurring in a time interval of length t has poisson distribution with parameter λt and the number of occurrences in non overlapping intervals are independent then the distribution of times between arrivals is exponential with parameter $\lambda = 1$

$$\alpha = \frac{\nu}{2} \quad \beta = 2$$

is called the "chi-squared"
 χ^2 distribution

$$f(x; \nu) = \begin{cases} \frac{1}{2^{\nu/2} \Gamma(\nu/2)} x^{(\nu/2)-1} e^{-x/2} & x > 0 \\ 0 & x < 0 \end{cases} \quad (5)$$

Note that ν is called the number of degrees of freedom (dof) of X

Note it can be shown that χ^2 distribution with dof ν is equivalent to the sum of ν ^{standard} normal r.v squared.

Examples

Suppose the time spent ^{by a student} on a computer per day is randomly distributed using the gamma distribution with mean 20 min, variance 80 min²

a) what is α and β ?

$$X \equiv \text{"time spent on computer"}$$

$$E[X] = \alpha\beta = 20$$

$$\text{Var}[X] = \alpha\beta^2 = 80 \Rightarrow$$

$$\Rightarrow \beta = 20/\alpha$$

$$\Rightarrow \frac{400}{\alpha} = 80$$

$$\Rightarrow \alpha = 5, \beta = 4$$

- b) What is the probability that the student uses at most 24 minutes on the computer?

$$P(X \leq 24) = \int_0^{24/4} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} dy$$

$$= .715 \quad (\text{based on table A.4 in the text})$$

- c) What is the probability that the student uses between 20 and 40 minutes on computer?

$$P(20 \leq X \leq 40) = \int_0^{10} \frac{1}{\Gamma(5)} y^{5-1} e^{-y} dy$$

$$- \int_0^5 \frac{1}{\Gamma(5)} y^{5-1} e^{-y} dy$$

$$= .971 - .560$$

$$= .411.$$