Lecture 21

Until now we have dealt mainly with situations where there is only 1 random variable. However, there are many situations where more than one random variable is of interest.

Joint probability density functions

Let X, Y be r.v. defined on the sample space S (or over the $\mathbb{R}^2$ space).

Discrete pairs of numbers (x, y)

$$p(x,y) = P(X = x \text{ and } Y = y)$$

Continuous

$$f(x,y) \leftarrow \text{density surface} \qquad \iint f(x,y)\,dx\,dy = 1$$

$$f(x,y) \geq 0$$

Probabilities

Let A be a set of (x,y) pairs

$$P((X,Y) \in A) = \sum_{(x,y)\in A}\sum p(x,y) \qquad \text{Discrete}$$

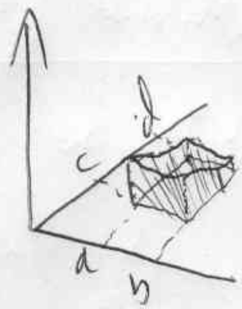
Let A be a two dimensional region

$$P((X,Y)\in A)=\iint_A f(x,y)\,dx\,dy \quad \text{Continuous}$$

ie probability is volume under densitysurface

For a rectangular region A

$$P((X,Y)\in A)=P(a\le X\le b,\ c\le Y\le d)=\int_a^b\int_c^d f(x,y)\,dy\,dx$$



Marginal probabilities

Suppose we have a joint pdf but want to know only pdf of one variable without reference to the other variable. How do we get it? by the marginal pdf

Discrete $p_X(x)=\sum_y p(x,y)$ $\quad p_Y(y)=\sum_x p(x,y)$

Continuous $f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy$ $\quad f_Y(y)=\int_{-\infty}^{\infty} f(x,y)\,dx$

Independence

Recall from an earlier lecture that if A and B are independent then $P(A \cap B) = P(A)P(B)$

By analogy If two r.v X and Y are independent for all (x, y) pairs then

$$p(x,y) = p_X(x)\, p_Y(y) \qquad \text{discrete}$$

$$f(x,y) = f_X(x)\, f_Y(y) \qquad \text{Continuous.}$$

~~Extending~~

Extending to more than two variables

All these definitions can be extended to more than two r.v. See your text book for more on this issue. The ideas are straightforward.

Conditional distributions

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} \quad \leftarrow \text{continuous} \quad -\infty < y < \infty$$

$$p_{Y|X}(y|X=x) = \frac{p(x,y)}{p_X(x)} \quad \leftarrow \text{discrete case.}$$

Some Examples

1. Suppose we have r.v X, Y with pdf

$$f(x,y)=\begin{cases} K(x^2+y^2) & 20\leq x\leq 30,\ 20\leq y\leq 30 \\ 0 & \text{otherwise} \end{cases}$$

a) for this to be valid pdf what is k?

To be valid need $\iint f(x,y)\,dx\,dy = 1$

so

$$\int_{20}^{30}\int_{20}^{30} K(x^2+y^2)\,dy\,dx$$

$$= K\int_{20}^{30}\left[x^2 y + \frac{y^3}{3}\right]_{y=20}^{y=30} dx$$

$$= K\int_{20}^{30}\left[30x^2 + 9000 - 20x^2 - \frac{8000}{3}\right]$$

$$= K\int_{20}^{30}\left[10x^2 + \frac{19000}{3}\right]dx$$

$$= K\left[\frac{10}{3}x^3 + \frac{19000}{3}x\right]\Bigg|_{x=20}^{x=30}$$

$$= K\left[90000 + 190000 - \frac{80000}{3} - \frac{380000}{3}\right]$$

$$= k\frac{380000}{3}$$

$$\Rightarrow k = \frac{3}{380000}$$

b) what is $P(X<26, Y<26)$?

$$P(X<26, Y<26) = \int_{20}^{26}\int_{20}^{26} \frac{3}{380000}(x^2+y^2)\,dy\,dx$$

$$= \frac{3}{380000}\int_{20}^{26}\left[x^2y+\frac{y^3}{3}\right]_{y=20}^{26} dx$$

$$= \frac{3}{380000}\int_{20}^{26}\left[26x^2+\frac{26^3}{3}-20x^2-\frac{20^3}{3}\right]dx$$

$$= \frac{3}{380000}\int_{20}^{26} 6x^2+\frac{26^3-20^3}{3}\,dx$$

$$= \frac{3}{380000}\left[\frac{6x^3}{3}+\frac{(26^3-20^3)x}{3}\right]_{20}^{26}$$

$$= \frac{1}{380000}\left[6(26)^3+(26^3-20^3)(26) - 6(20)^3-(26^3-20^3)(20)\right]$$

$$= \frac{1}{380000}\left[6(26^3-20^3) + 6(26^3-20^3)\right]$$

$$= \frac{12(26^3-20^3)}{380000} = .3024$$

c) whats the ~~[illegible]~~ marginal distribution of X?

$$f_X(x) = \int_{20}^{30} \frac{3}{380000}(x^2+y^2)\,dy$$

$$= \frac{3}{380000}\left[x^2 y + \frac{y^3}{3}\right]\Bigg|_{y=20}^{y=30}$$

$$= \frac{3}{380{,}000}\left[10x^2 + \frac{30^3-20^3}{3}\right]$$

d) whats the marginal distribution of Y?

$$f_Y(y) = \int_{20}^{30} \frac{3}{380000}(x^2+y^2)\,dx$$

$$= \frac{3}{380000}\left[\frac{x^3}{3} + y^2 x\right]\Bigg|_{x=20}^{x=30}$$

$$= \frac{3}{380000}\left[10y^2 + \frac{30^3-20^3}{3}\right]$$

e) Are X, Y independent?

need $f(x,y) = f_X(x) f_Y(y)$

$$\frac{3}{380000}\left[10x^2 + \frac{30^3-20^3}{3}\right]\left[\frac{3}{380000}\left(10y+\frac{30^3-20^3}{3}\right)\right]$$

$$\neq \frac{3}{380000}\left[x^2+y^2\right]$$

So Not independent.

f) Whats probability $Y \leq 26$ given that $X=30$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

$$= \frac{\cancel{k}(x^2+y^2)}{\cancel{k}\left[10x^2+\frac{30^3-20^3}{3}\right]}$$

with $\cancel{X} = x = 30$

$$f_{Y|X}(y|30) = \frac{30^2+y^2}{\left[10(30)^2+\frac{30^3-20^3}{3}\right]} = \frac{3}{46000}(30^2+y^2)$$

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$$P(Y \le 26 \mid X=30) = \int_{20}^{26} f_{Y|X}(y|30)\,dy$$

$$= \frac{3}{46000}\int_{20}^{26}(30^2+y^2)\,dy$$

$$= \frac{3}{46000}\left[30^2 y + \frac{y^3}{3}\right]_{20}^{26}$$

$$= \frac{3}{46000}\left[30^2(6) + \frac{26^3-20^3}{3}\right]$$

$$= .5603$$

9) what is

$P(|X-Y| \le 2)$?

First consider what the condition implies about Y in terms of X. For any fixed $X=x$ it must be true that $x-2 \le y \le x+2$. And we allow that X ranges from 20 to 30

so

$$P(|X-Y| \le 2) = \int_{20}^{30}\int_{x-2}^{x+2} \frac{3(x^2+y^2)}{380000}\,dy\,dx$$

$$= \int_{20}^{30} \frac{3}{380000} \left[x^2 y + \frac{y^3}{3} \right]_{x-2}^{x+2} dx$$

$$= \int_{20}^{30} \frac{3}{380000} \left[\begin{array}{l} x^2(x+2) + \frac{(x+2)^3}{3} \\ - x^2(x-2) - \frac{(x-2)^3}{3} \end{array} \right] dx$$

$$= \frac{3}{380000} \int_{20}^{30} \left[4x^2 + \frac{(x^3+6x^2+12x+8 - (x^3-6x^2+12x-8))}{3} \right] dx$$

$$= \frac{3}{380000} \int_{20}^{30} 4x^2 + \frac{12x^2+16}{3} \, dx$$

$$= \frac{3}{380000} \int_{20}^{30} \frac{24x^2+16}{3} \, dx$$

$$= \frac{1}{380000} \left[\frac{24x^3}{3} + 16x \right]_{20}^{30}$$

$$= \frac{1}{380000} \left[\frac{24}{3}(30^3-20^3) + 16(30-20) \right]$$

$$= .4004 \ (4dp)$$