

# Lecture 28

①

## A few more MLE examples

### Example 1

Suppose we <sup>take a</sup> sample of size  $n$  from a distribution with pdf

$$f(x; \theta) = \begin{cases} 2\theta x^{-3} & \theta \leq x \\ 0 & x < \theta, 0 < \theta \end{cases}$$

Likelihood

$$L(\theta) = \prod_{i=1}^n 2\theta^2 x_i^{-3} = 2^n \theta^{2n} \prod_{i=1}^n x_i^{-3}$$

Log likelihood

$$l(\theta) = \log L(\theta) = n \log 2 + 2n \log \theta - 3 \sum_{i=1}^n \log x_i$$

$$\frac{\partial l(\theta)}{\partial \theta} = \frac{2n}{\theta} = 0?$$

Calculus doesn't help us here. However, by reasoning it seems that since  $\theta \leq x$  and the pdf is 0 when  $x < \theta$

$$\text{and } f(x; \theta) = 2\theta^2 x^{-3}$$

②

it makes sense to choose  $\hat{\theta} = \min(x_i) = x_{[1]}$

↑  
notation for  
order statistic.

### Example 2

A random sample of size  $n$  from

$$f(x; \theta_1, \theta_2) = \frac{1}{\theta_2 - \theta_1} \quad \theta_1 \leq x \leq \theta_2$$

Again calculus will not help us. But reasoning says that making the difference between  $\theta_2$  and  $\theta_1$  as small as possible will maximize the likelihood. since all  $x_i$  must satisfy

$$\theta_1 \leq x_i \leq \theta_2$$

it stands to reason that we should take

$$\hat{\theta}_1 = \min x_i = x_{[1]} \quad \text{and} \quad \hat{\theta}_2 = \max x_i = x_{[n]}$$

### Example 3

(3)

Consider independent random samples  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_m$  from normal distributions with common mean  $\mu$  but different variances  $\sigma_1^2$  and  $\sigma_2^2$ . What are the MLE of  $\mu, \sigma_1^2$  and  $\sigma_2^2$ ?

Likelihood

$$L(\theta) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{1}{2\sigma_1^2}(X_i - \mu)^2\right) \prod_{j=1}^m \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{1}{2\sigma_2^2}(Y_j - \mu)^2\right)$$

$$\log L(\theta) = \frac{1}{(\sqrt{2\pi})^n (\sigma_1^2)^{\frac{n}{2}}} \exp\left(-\frac{1}{2\sigma_1^2} \sum_{i=1}^n (X_i - \mu)^2\right) \frac{1}{(\sqrt{2\pi})^m (\sigma_2^2)^{\frac{m}{2}}} \exp\left(-\frac{1}{2\sigma_2^2} \sum_{j=1}^m (Y_j - \mu)^2\right)$$

Log likelihood

$$l(\theta) = \frac{-(n+m)}{2} \log(2\pi) - \frac{n}{2} \log \sigma_1^2 - \frac{m}{2} \log \sigma_2^2 - \frac{1}{2\sigma_1^2} \sum_{i=1}^n (X_i - \mu)^2 - \frac{1}{2\sigma_2^2} \sum_{j=1}^m (Y_j - \mu)^2$$

$$\frac{\partial l}{\partial \mu} = \frac{\sum_{i=1}^n (x_i - \hat{\mu})}{\hat{\sigma}_1^2} + \frac{\sum_{j=1}^m (y_j - \hat{\mu})}{\hat{\sigma}_2^2} = 0 \quad (1) \quad \textcircled{+}$$

$$\frac{\partial l}{\partial (\sigma_1^2)} = -\frac{n}{2\hat{\sigma}_1^2} + \frac{1}{2(\hat{\sigma}_1^2)^2} \sum_{i=1}^n (x_i - \hat{\mu})^2 = 0 \quad (2)$$

$$\frac{\partial l}{\partial (\sigma_2^2)} = -\frac{m}{2\hat{\sigma}_2^2} + \frac{1}{2(\hat{\sigma}_2^2)^2} \sum_{j=1}^m (y_j - \hat{\mu})^2 = 0 \quad (3)$$

from (1)

$$\Rightarrow \frac{\sum x_i - n\hat{\mu}}{\hat{\sigma}_1^2} + \frac{\sum y_j - m\hat{\mu}}{\hat{\sigma}_2^2} = 0$$

$$\hat{\sigma}_2^2 (\sum x_i - n\hat{\mu}) + \hat{\sigma}_1^2 (\sum y_j - m\hat{\mu}) = 0$$

$$\Rightarrow \hat{\mu} = \frac{\hat{\sigma}_2^2 \sum x_i + \hat{\sigma}_1^2 \sum y_j}{n\hat{\sigma}_2^2 + m\hat{\sigma}_1^2} \quad (4)$$

(5)

from (2)

$$\hat{\sigma}_1^2 = \frac{\sum_{i=1}^n (X_i - \hat{\mu})^2}{n} \quad (5)$$

from (3)

$$\hat{\sigma}_2^2 = \frac{\sum_{j=1}^m (Y_j - \hat{\mu})^2}{m} \quad (6)$$

So replacing these into (4) gives

$$\hat{\mu} = \frac{\sum_{j=1}^m (Y_j - \hat{\mu})^2}{m} \sum_{i=1}^n X_i + \frac{\sum_{i=1}^n (X_i - \hat{\mu})^2}{n} \sum_{j=1}^m Y_j$$


---


$$\hat{\mu} = \frac{n \sum_{j=1}^m (Y_j - \hat{\mu})^2}{m} + \frac{m \sum_{i=1}^n (X_i - \hat{\mu})^2}{n}$$

Solve this for  $\hat{\mu}$  and re-substitute it into (5) and (6) for the solution.