Lecture 37

Today we address the two sample t-test and confidence intervals for differences between means with σ_1, σ_2 unknown.

Framework

Assume that $X_1, \ldots, X_m$ is a random sample from $N(\mu_1, \sigma_1)$ and that $Y_1, \ldots, Y_n$ is an independent random sample from $N(\mu_2, \sigma_2)$. If this is the case then the standardized r.v.

$$T = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}}$$

has approximately a t distribution with df ν

where $$\nu = \frac{\left(\frac{S_1^2}{m} + \frac{S_2^2}{n}\right)^2}{\frac{(S_1^2/m)^2}{m-1} + \frac{(S_2^2/n)^2}{n-1}}$$ rounded down to nearest integer.

Confidence Interval

Based on this statistic it can be shown that a $100(1-\alpha)\%$ CI for $\mu_1 - \mu_2$ is given by

$$\bar{x} - \bar{y} \pm t_{\alpha/2, \nu} \sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}$$

Hypothesis test

Similarly using ideas discussed previously to test

$H_0: \mu_1 - \mu_2 = \Delta_0$ against any of the alternatives

use the test statistic

$$t = \frac{\bar{x} - \bar{y} - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}}$$

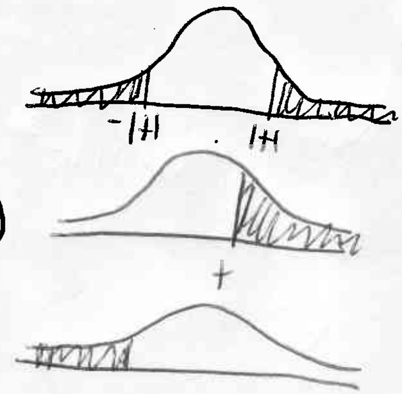
So P-values come from t-table with $df = \nu$

As previously discussed the particular region to use for the P-value depends on the alternative

$H_A: \mu_1 - \mu_2 \neq \Delta_0$ $\quad 2P(T > |t|)$

$H_A: \mu_1 - \mu_2 > \Delta_0$ $\quad P(T > t)$

$H_A: \mu_1 - \mu_2 < \Delta_0$ $\quad P(T < t)$



Pooled 2 sample procedures

If we are willing to introduce the further assumption that $\sigma_1^2 = \sigma_2^2 = \sigma^2$ (ie that the true variances are equal) then we may use some alternative techniques. In particular the first thing we need is an estimator of σ^2. what we do is take information from both samples and combine it together (this is where "pooling" comes from). In particular the estimator is

$$S_p^2 = \frac{m-1}{m+n-2} S_1^2 + \frac{n-1}{m+n-2} S_2^2$$

Therefore our standardized statistic becomes.

$$T = \frac{(\bar{x}-\bar{y})-(\mu_1-\mu_2)}{\sqrt{\frac{S_p^2}{m}+\frac{S_p^2}{n}}} = \frac{(\bar{x}-\bar{y})-(\mu_1-\mu_2)}{S_p\sqrt{\frac{1}{m}+\frac{1}{n}}}$$

and it can be shown that this statistic has the t-distribution with exactly $df = m+n-2$

Pooled 2 sample CI

using this new statistic it can be shown that a $100(1-\alpha)\%$ CI for $\mu_1-\mu_2$ is given by

$$\bar{x}-\bar{y} \pm t_{\alpha/2,m+n-2}\, S_p\sqrt{\frac{1}{m}+\frac{1}{n}}$$

Pooled 2 sample hypothesis test

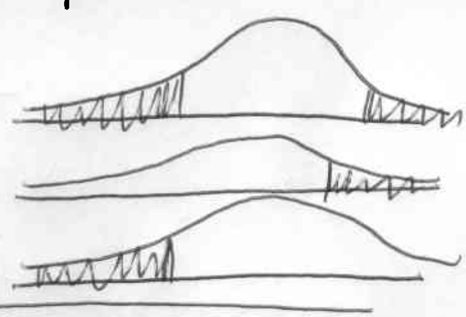
To test the hypothesis

$H_0: \mu_1 - \mu_2 = \Delta_0$ against any of the alternatives

use

$$t = \frac{\bar{x} - \bar{y} - \Delta_0}{s_p\sqrt{\frac{1}{m} + \frac{1}{n}}}$$

So P-values come from the t-table. As always look at alternative hypothesis to figure out which region to use.



$H_A: \mu_1 - \mu_2 \neq \Delta_0$ — $2P(T > |t|)$

$H_A: \mu_1 - \mu_2 > 0$ — $P(T > t)$

$H_A: \mu_1 - \mu_2 < 0$ — $P(T < t)$

Examples

1. An article on the "Flexure of Concrete Beams reinforced with Advanced Composite Orthogrids" (J. Aerospace Engr., 1997, 7-15) gave the following data for the Ultimate load in kN for

two different types of beams

Type	Samp Size	Samp Mean	Samp SD
Fiberglass grid	26	33.4	2.2
Carbon Grid	26	42.8	4.3

Assume normality and then find a 99% CI for the difference in means. Then test whether the Carbon Grid have higher mean ultimate load than the fiberglass grid beams.

Let μ_1 = mean ultimate load for Fiberglass grid

μ_2 = mean " " " Carbon grid

what df?

$$df = \frac{\left(\frac{2.2^2}{26} + \frac{4.3^2}{26}\right)^2}{\frac{(2.2^2/26)^2}{25} + \frac{(4.3^2/26)^2}{25}} = 28.03$$

so use $\nu = 28$ degrees of freedom

So 99% CI is given by

$$33.4 - 42.8 \pm 2.763 \sqrt{\frac{2.2^2}{26} + \frac{4.3^2}{26}}$$

$$-9.4 \pm 4.97$$

So 99% CI for $\mu_1 - \mu_2$ is

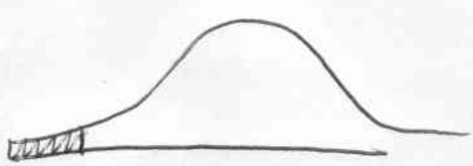
$$(-14.37, -4.43)$$

$H_0: \mu_1 - \mu_2 \geq 0$

$H_A: \mu_1 - \mu_2 < 0$

test statistic

$$t = \frac{33.4 - 42.8}{\sqrt{\frac{2.2^2}{26} + \frac{4.3^2}{26}}} = -5.219$$



$$\text{Pvalue} = P(T < -5.219)$$
$$\leq P(T < -3.674)$$
$$= .0005$$

Since P-value is very small plenty of evidence against H_0.

Therefore we conclude that mean ultimate load for carbon grid is higher.