Lecture 39

Today we discuss two sample proportion problems.

Framework

Suppose that $X \sim Bin(m, p_1)$ and $Y \sim Bin(n, p_2)$. with X, Y being independent. Let $\hat{p}_1 = \frac{X}{m}$ and $\hat{p}_2 = \frac{Y}{n}$. Then $\hat{p}_1 - \hat{p}_2$ is an unbiased estimator of $p_1 - p_2$ (ie $E[\hat{p}_1 - \hat{p}_2] = p_1 - p_2$).

Also $Var(\hat{p}_1 - \hat{p}_2) = \frac{p_1(1-p_1)}{m} + \frac{p_2(1-p_2)}{n}$.

Recall that if m is large then $\hat{p}_1$ has approximately Normal distribution and that if n is large then $\hat{p}_2$ has approximately normal distribution. This implies that $\hat{p}_1 - \hat{p}_2$ will also have approximately Normal distribution. Therefore the ~~test~~ statistic

$$Z = \frac{\hat{p}_1 - \hat{p}_2 - (p_1 - p_2)}{\sqrt{\frac{p_1(1-p_1)}{m} + \frac{p_2(1-p_2)}{n}}}$$

will have approximately $N(0,1)$ distribution.

<u>Confidence interval for $p_1 - p_2$</u>

Supposing that we replace the sd term in previous statistic ~~result~~ with the corresponding SE, then using the previously described procedure it is easy to show that a $100(1-\alpha)\%$ CI for $p_1 - p_2$ is given by

$$\hat{p}_1 - \hat{p}_2 \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{m} + \frac{\hat{p}_2(1-\hat{p}_2)}{n}}$$

<u>Hypothesis testing</u>

Suppose that we want to test hypotheses of the form

$$H_0: p_1 - p_2 = 0$$

vs some alternative. A reasonable assumption for the previously mentioned ~~test~~ statistic under this null hypothesis is to take $p_1 = p_2 = p$ in the ~~std~~ part of the statistic. Therefore when H_0 is true

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{p(1-p)\left(\frac{1}{m} + \frac{1}{n}\right)}}$$

has approximately $N(0,1)$ distribution. what do we need to use this as a test statistic? an estimate $\hat{p}$ of p.

Similar to the pooled two sample situation with means we take a weighted average of p_1 and p_2.

In particular

$$\hat{p} = \frac{m}{m+n}\hat{p}_1 + \frac{n}{m+n}\hat{p}_2 = \frac{X+Y}{m+n}$$

So to test

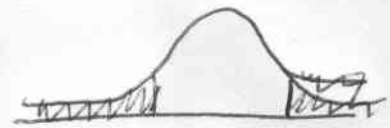
$H_0: p_1 - p_2 = 0$

against any of the alternatives use the test statistic

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n} + \frac{1}{m}\right)}}$$

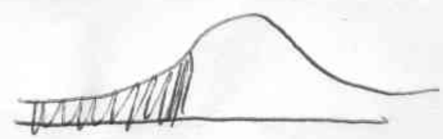
As per usual the P-value is determined by the alternative

$H_A: p_1 - p_2 \neq 0$ $\quad 2P(Z > |z|)$



$H_A: p_1 - p_2 > 0$ $\quad P(Z > z)$

$H_A: p_1 - p_2 < 0$ $\quad P(Z < z)$



Example

Gastric freezing was once a recommended treatment for ulcers in the upper intestine. A randomized comparative experiment found that 28 of 82 patients improved when subjected to gastric freezing while 30 of 78 patients in a control group improved.

Use a hypothesis test to check whether gastric freezing is more likely to improve a patients condition.

Let p_1 = proportion of patients who are gastric frozen that improve condition

p_2 = proportion of control patients who improve

$\hat{p}_1 = \frac{28}{82} = 0.3414$ $\qquad$ $\hat{p}_2 = \frac{30}{78} = 0.3846$

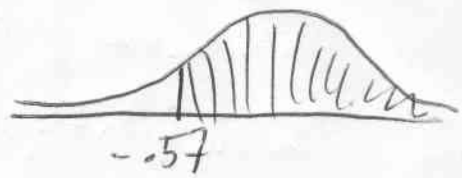
$\hat{p} = \frac{28+30}{82+78} = 0.3625$

$H_0: p_1 \leq p_2$ ie $p_1 - p_2 \leq 0$

$H_A: p_1 > p_2$ ie $p_1 - p_2 > 0$

$$z = \frac{.3414 - .3846}{\sqrt{.3625(1-.3625)\left(\frac{1}{82}+\frac{1}{78}\right)}} = -.568$$

$$P\text{value} = P(Z > -.57)$$



$$= 1 - \Phi(-.57)$$

$$= 1 - .2843$$

$$= .7157$$

Since this P-value is large cannot reject the null hypothesis, in fact it appears that gastric surgery might even hinder rather than help a patient to improve.

The relationship between tests and CI

Although we have not explictly stated, it to this point there is a relationship between CI and the result of a hypothesis test. In particular the results of a significance level α hypothesis test can ~~bee~~ related to a corresponding

$100(1-\alpha)\%$ Confidence Interval.

Eg

Test	CI
$H_0: \mu = \mu_0$	$\bar{x} \pm t_{\alpha/2, n-1} \frac{s}{\sqrt{n}}$
$H_A: \mu \neq \mu_0$	
$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$	

At level of significance α a rejection of the H_0 is equivalent to μ_0 not being inside the $100(1-\alpha)\%$ CI. Similarly failure to reject H_0 is equivalent to μ_0 being inside the interval.
ie we can determine the results of the hypothesis test using the CI.