Lecture 9

Last time we introduced counting techniques. Lets combine these techniques with our statements about probability for equally likely outcomes ie $P(A) = \frac{N(A)}{N}$

Example 1

A box contains 8 red, 3 white and 9 blue balls. If 3 balls are drawn at random without replacement determine

(a) P(all 3 are red)

number of ways to pick 3 out of 8 red balls

$$ {}_8C_3 = \frac{8!}{5!3!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56 $$

number of ways to pick 3 balls out of 20

$$ {}_{20}C_3 = \frac{20!}{17!3!} = \frac{20 \times 19 \times 18}{3 \times 2 \times 1} = 1140 $$

$$ \Rightarrow P(\text{all 3 are red}) = \frac{56}{1140} = \frac{14}{285} $$

(b) P(2 are red and 1 is white)

ways to pick red balls $^{8}C_2 = \frac{8!}{6!2!} = 28$

ways to pick white ball $^{3}C_1 = \frac{3!}{2!1!} = 3$

$\Rightarrow$ P(2 are red and 1 is white) =

$$\frac{28 \times 3}{1140} = \frac{84}{1140} = \frac{21}{285} = \frac{7}{95}$$

(c) $P(1 \text{ of each color}) = \frac{^{8}C_1 \, ^{3}C_1 \, ^{9}C_1}{^{20}C_3}$

$$= \frac{8 \times 3 \times 9}{1140} = \frac{216}{1140} = \frac{54}{285} = \frac{18}{95}$$

Example 2

Poker involves 5 cards drawn from 52 shuffled cards

(a) $P(4 \text{ aces}) = \dfrac{({}^{4}C_{4})({}^{48}C_{1})}{{}^{52}C_{5}} = \dfrac{\frac{\cancel{4!}}{0!\cancel{4!}} \; \frac{48!}{47!1!}}{\frac{52!}{47!5!}} = \dfrac{48 \times 5!}{52 \times 51 \times 50 \times 49 \times 48}$

$= \dfrac{5760}{31187520}$

(b) $P(4 \text{ aces and } 1 \text{ King}) = \dfrac{({}^{4}C_{4})({}^{4}C_{1})}{{}^{52}C_{5}}$

(c) $P(3 \text{ are 10s and 2 are J}) = \dfrac{({}^{4}C_{3})({}^{4}C_{2})}{{}^{52}C_{5}}$

choose first suit choose second suit

(d) $P(3 \text{ of one suit, 2 of another}) = \dfrac{(4 * {}^{13}C_{3})(3 * {}^{13}C_{3})}{{}^{52}C_{5}}$

(e) $P(\text{at least one ace}) = 1 - P(\text{no ace})$

$P(\text{no ace}) = \dfrac{{}^{48}C_{5}}{{}^{52}C_{5}}$

Conditional Probability

Notation $P(A \mid B)$

"conditional probability of A given that event B has occured"

The idea is that it is possible that the outcome of one event may change the probability of a subsequent event.

Formula For any two events A and B, $P(B) > 0$

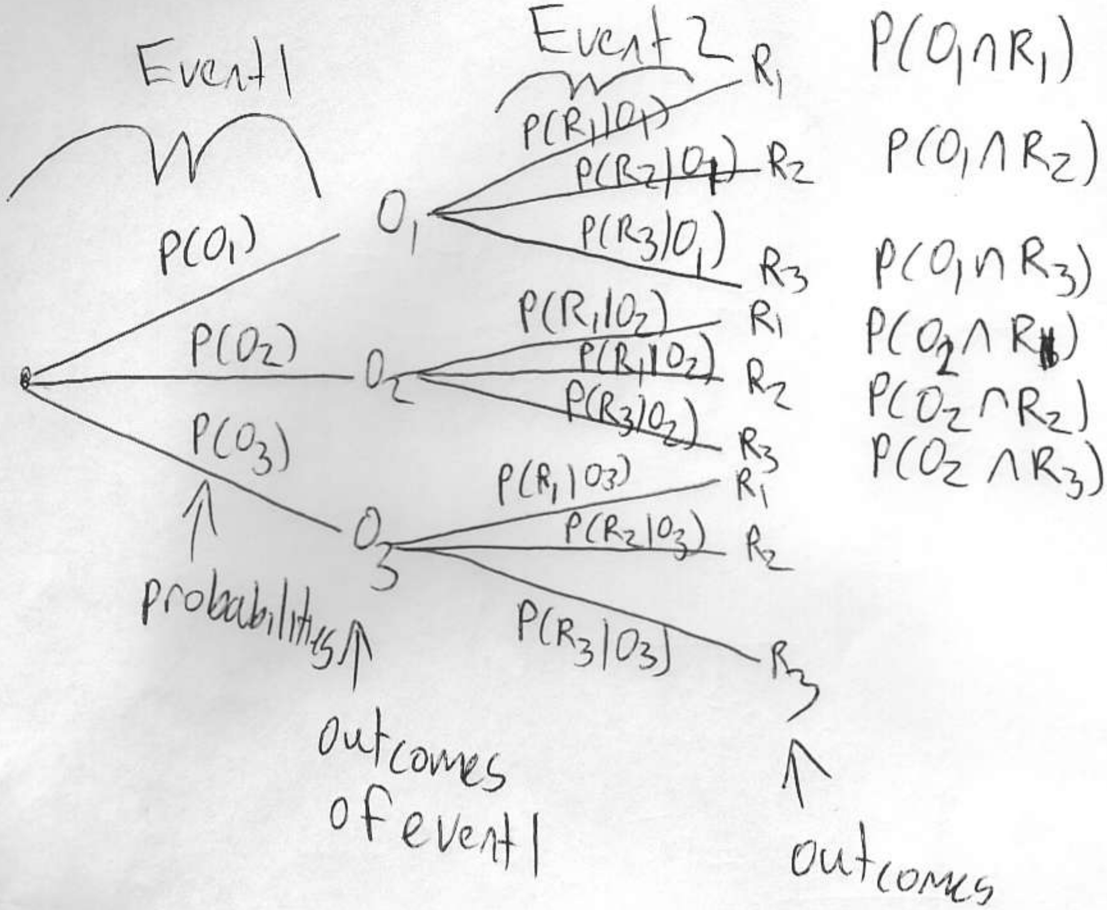
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Note this formula implies

$$P(A \cap B) = P(A \mid B)P(B)$$

"multiplication rule"

Tree Diagrams



Bayes Theorem

Suppose that $A_1, \ldots, A_K$ are mutually exclusive and exhaustive. Then For any event B

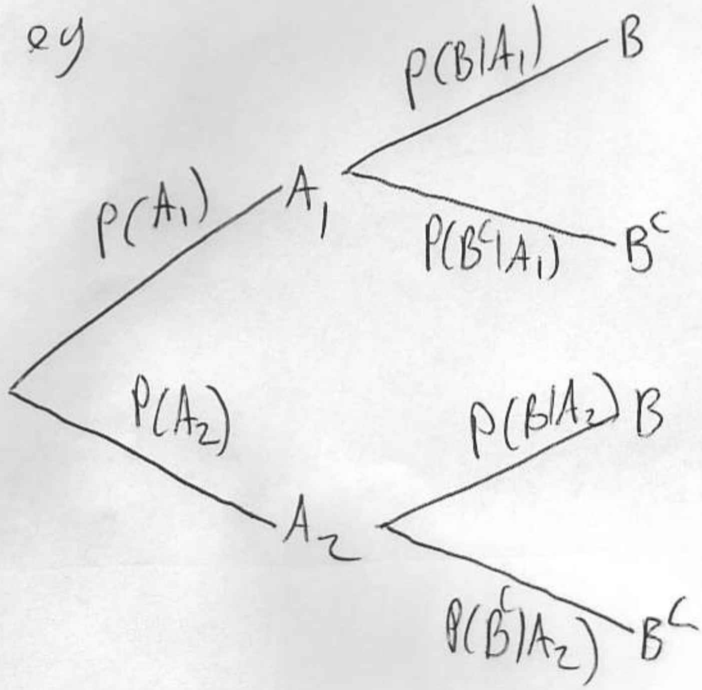
$$P(B) = \sum_{i=K}^{K} P(B|A_i)P(A_i) \leftarrow \text{The Law of total probability}$$

Introduce the condition $P(A_i) > 0 \quad i=1, \ldots k$
and $P(B)$ to above - Then

$$\boxed{P(A_j \mid B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B \mid A_j)P(A_j)}{\sum_{i=1}^{k} P(B \mid A_i)P(A_i)} \qquad j = 1, \ldots k}$$

Bayes Theorem

eg



$$P(A_1 \mid B) = \frac{P(B \mid A_1)P(A_1)}{P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2)}$$

Example

In a bolt factory machines 1, 2, 3 produce 20%, 30% or 50% of the output.
OF ~~there~~ respective outputs 5%, 3% and 2% are defective. A bolt is selected at random.

(a) P(bolt is defective)

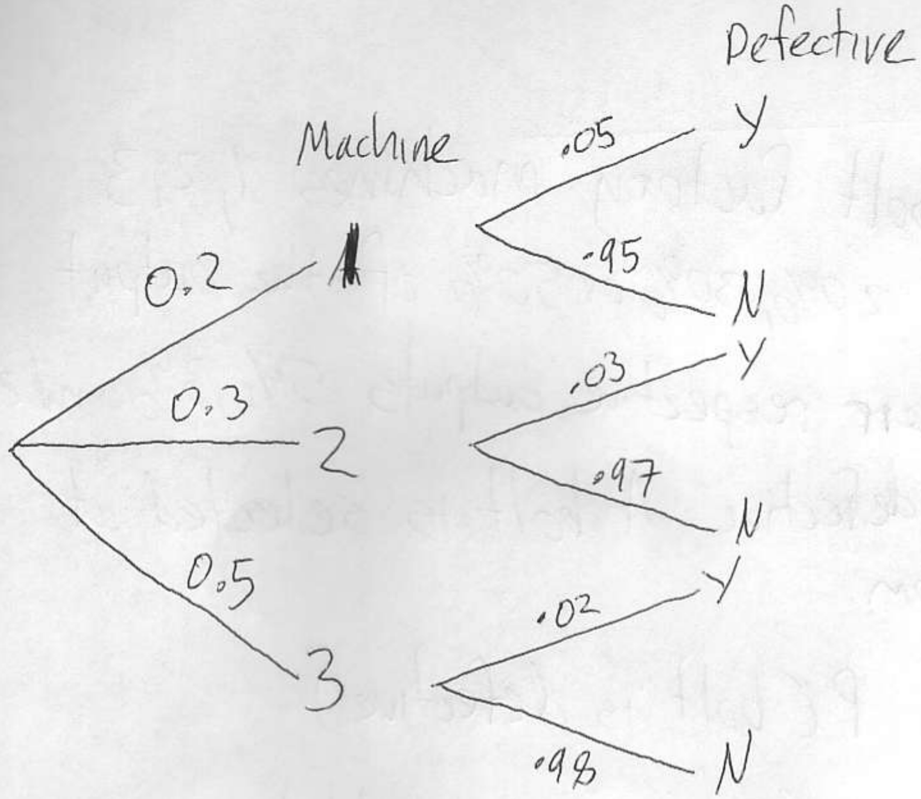
= P(Machine 1 and defective)
+ P(Machine 2 and defective)
+ P(machine 3 and defective)

(b) P(From machine 1 | defective)

$$= \frac{P(\text{machine } 1 \cap \text{defective})}{P(\text{Defective})}$$

How do we get these probabilities?

Use tree diagram to make it easier to see.



$$P(\text{Defective}) = 0.2 * 0.05 + 0.3 * .03 + .5 \times .02$$
$$= 0.029$$

$$P(\text{machine } 1 \cap \text{Defective}) = 0.2 \times 0.05$$
$$= 0.01$$

$$\Rightarrow P(\text{machine } 1 \mid \text{Defective}) = \frac{0.01}{0.029} = 0.3448 \text{ (4dp)}$$