

Homework 3 Solutions

CH3: 13, 22, 31, 37, 52, 62, 64, 82

CH4: 8, 11

Problem 13

$$\begin{aligned} \text{a) } P(X \leq 3) &= P(X=0) + P(X=1) + P(X=2) + P(X=3) \\ &= .1 + .15 + .2 + .25 \\ &= .7 \end{aligned}$$

$$\begin{aligned} \text{b) } P(X < 3) &= P(X=0) + P(X=1) + P(X=2) \\ &= .1 + .15 + .2 \\ &= .45 \end{aligned}$$

$$\begin{aligned} \text{c) } P(X \geq 3) &= P(X=3) + P(X=4) + P(X=5) + P(X=6) \\ &= .25 + .20 + .06 + .04 \\ &= .55 \end{aligned}$$

$$\text{or } P(X \geq 3) = 1 - P(X < 3) = 1 - .45 = .55$$

$$\begin{aligned} \text{d) } P(2 \leq X \leq 5) &= P(X=2) + P(X=3) + P(X=4) + P(X=5) \\ &= .20 + .25 + .20 + .06 \\ &= .71 \end{aligned}$$

e)

X = "number of lines in use"

$\Rightarrow 6 - X$ = "number of lines not in use"

$$2 \leq 6 - X \leq 4 \Rightarrow 4 \geq X \geq 2$$

$$\begin{aligned} P(2 \leq X \leq 4) &= P(X=2) + P(X=3) + P(X=4) \\ &= .2 + .25 + .2 = .65 \end{aligned}$$

②

f) $6 - x \geq 4 \Rightarrow x \leq 2$
 $P(X \leq 2) = .45$ (from b)

Problem 22

a) $p(2) = P(X=2) = F(2) - F(1)$
 $= .39 - .19$
 $= .2$

b) $P(X > 3) = 1 - P(X \leq 3)$
 $= 1 - F(3)$
 $= 1 - .67$
 $= .33$

c) $P(2 \leq X \leq 5) = F(5) - F(1)$
 $= .97 - .19$
 $= .78$

d) $P(2 < X < 5) = P(3 \leq X \leq 4)$
 $= F(4) - F(2)$
 $= .92 - .39$
 $= .53$

Problem 31

a) $E(X) = \sum x_i P(X=x_i) = 13.5(.2) + 15.9(.5) + 19.1(.3)$
 $= 16.38$

③

$$E(X^2) = \sum_i x_i^2 P(X=x_i) = (13.5)^2(.2) + (15.9)^2(.5) + (19.1)^2(.3)$$

$$= 272.298$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 272.298 - (16.38)^2 = 3.9936$$

$$\text{b) } E[25X - 8.5] = 25E[X] - 8.5$$

$$= \del{16.38 \cdot 25 - 8.5} 401$$

$$\text{c) } \text{Var}(25X - 8.5) = 25^2 \text{Var}(X)$$

$$= 2496$$

$$\text{d) } E[h(X)] = E[X - .01X^2] = E[X] - .01E[X^2]$$

$$= 16.38 - .01(272.298) = 13.6570$$

(check) $E[X - .01X^2] = \sum_i (x_i - .01x_i^2) p(x_i)$

$$= \sum_i x_i p(x_i) - \sum_i .01x_i^2 p(x_i)$$

$$= E[X] - .01E[X^2]$$

Problem 37

$$E\left[\frac{1}{X}\right] = 1\left(\frac{1}{6}\right) + \frac{1}{2}\left(\frac{1}{6}\right) + \frac{1}{3}\left(\frac{1}{6}\right) + \frac{1}{4}\left(\frac{1}{6}\right) + \frac{1}{5}\left(\frac{1}{6}\right) + \frac{1}{6}\frac{1}{6}$$

$$= \frac{1}{6} + \frac{1}{12} + \frac{1}{18} + \frac{1}{24} + \frac{1}{30} + \frac{1}{36} = .40933 > 0.2857 = \frac{1}{3.5}$$

Problem 52

$X =$ "number of students in sample who received special accommodations"

$$X \sim \text{Bin}(25, .02)$$

a) $P(X=1) = \binom{25}{1} \cdot .02^1 \cdot .98^{24} = .3079$

b) $P(X \geq 1) = 1 - P(X < 1)$
 $= 1 - (P(X=0))$
 $= 1 - \left(\binom{25}{0} \cdot .02^0 \cdot .98^{25} \right)$
 $= 1 - .6035 = .3965$

c) $P(X \geq 2) = 1 - P(X \leq 1)$
 $= 1 - (P(X=0) + P(X=1))$
 $= 1 - .3079 - .6035 = .0986$

d) $E[X] = 25(.02) = .5$

$$SD(X) = \sqrt{25(.02)(.98)} = \sqrt{.49} = .7$$

$$.5 \pm 2(.7) = (-0.9, 1.9)$$

so within 2SD of expected number is $X=0$ or $X=1$

$$P(X=0) + P(X=1) = 1 - P(X \geq 2) = 1 - .0986 = .9014$$

e) "number of hours allowed" $\equiv Y$
for all students

$$Y = 4.5X + 3(25 - X)$$

$$= 1.5X + 75$$

$$\Rightarrow E[Y] = E[1.5X + 75]$$

$$= 1.5E[X] + 75$$

$$= 1.5(5) + 75$$

$$= 75.75$$

\Rightarrow number of hours allowed $\frac{75.75}{25} = 3.03$ hours

Problem 62

a) X = "number of reservations who show up"

$$X \sim \text{Bin}(6, 0.8)$$

At least one individual cannot be accommodated

$$\Rightarrow X=5 \text{ or } X=6 \text{ i.e. } X > 4$$

$$P(X > 4) = P(X=5) + P(X=6)$$

$$= \binom{6}{5} \cdot 0.8^5 \cdot 0.2^1 + \binom{6}{6} \cdot 0.8^6 \cdot 0.2^0 = 0.6554$$

b) "number of places available" $= 4 - X$

$$E[4 - X] = 4 - E[X] = 4 - 6(0.8) = -0.8$$

(C) X = "number of passengers on trip"

$$P(X=x) = \sum_{y=3}^6 P(x \text{ turned up and } y \text{ reserved})$$

$$= \sum_{y=3}^6 P(x \text{ turned up} \mid y \text{ reserved}) P(y \text{ reserved})$$

So

$$P(X=0) = P(0 \text{ turned up and } 3 \text{ reserved})$$

$$+ P(0 \text{ turned up and } 4 \text{ reserved})$$

$$+ P(0 \text{ turned up and } 5 \text{ reserved})$$

$$+ P(0 \text{ turned up and } 6 \text{ reserved})$$

$$= \binom{3}{0} \cdot 8^0 \cdot 2^3 (.1) + \binom{4}{0} \cdot 8^0 \cdot 2^4 (.2) + \binom{5}{0} \cdot 8^0 \cdot 2^5 (.3)$$

$$+ \binom{6}{0} \cdot 8^0 \cdot 2^6 (.4) = .0012$$

$$P(X=1) = \binom{3}{1} \cdot 8^1 \cdot 2^2 (.1) + \binom{4}{1} \cdot 8^1 \cdot 2^3 (.2) + \binom{5}{1} \cdot 8^1 \cdot 2^4 (.3)$$

$$+ \binom{6}{1} \cdot 8^1 \cdot 2^5 (.4)$$

$$= .0173$$

$$P(X=2) = \binom{3}{2} \cdot 8^2 \cdot 2^1 (0.1) + \binom{4}{2} \cdot 8^2 \cdot 2^2 (0.2) + \binom{5}{2} \cdot 8^2 \cdot 2^3 (0.3) \\ + \binom{6}{2} \cdot 8^2 \cdot 2^4 (0.4) = .0906$$

$$P(X=3) = \binom{3}{3} \cdot 8^3 \cdot 2^0 (0.1) + \binom{4}{3} \cdot 8^3 \cdot 2^1 (0.2) + \binom{5}{3} \cdot 8^3 \cdot 2^2 (0.3) \\ + \binom{6}{3} \cdot 8^3 \cdot 2^3 (0.4) = .2274$$

$$P(X=4) = 1 - (P(X=0) + P(X=1) + P(X=2) + P(X=3)) \\ = 1 - (.0012 + .0173 + .0906 + .2274) \\ = ~~0.6636~~ .6636$$

Problem 64

8

a) $X \sim \text{Hyp}(5, 6, 15)$

↑ sample size ↖ population size
↙ # 3 megapixels in population

$$b) P(X=2) = \frac{\binom{6}{2} \binom{9}{3}}{\binom{15}{5}} = \frac{\frac{6!}{2!4!} \frac{9!}{3!6!}}{\frac{15!}{5!10!}} = .290$$

~~●~~ $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$$= \frac{\binom{6}{0} \binom{9}{5}}{\binom{15}{5}} + \frac{\binom{6}{1} \binom{9}{4}}{\binom{15}{5}} + \frac{\binom{6}{2} \binom{9}{3}}{\binom{15}{5}} = .573$$

$$\begin{aligned} P(X > 2) &= 1 - P(X \leq 2) \\ &= 1 - (P(X \leq 2) - P(X=2)) \\ &= 1 - (.573 - .290) = .707 \end{aligned}$$

①

$$c) E[X] = 5 \left(\frac{6}{15} \right) = 2$$

$$SD(X) = \sqrt{\left(\frac{15-5}{15-1} \right) 5 \frac{6}{15} \left(1 - \frac{6}{15} \right)} = \sqrt{.957} = .978$$

Problem 52

$X =$ "number of people who arrive" ^{in an hour}
 $X \sim \text{Poi}(5)$

$$a) P(X=4) = \frac{e^{-5} 5^4}{4!} = .1755$$

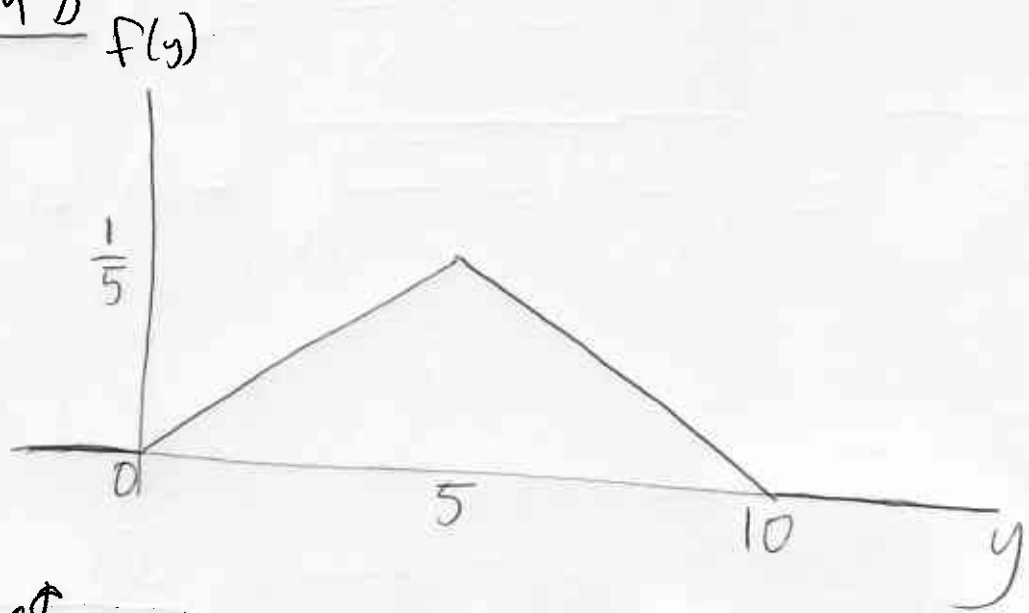
$$\begin{aligned} b) P(X \geq 4) &= 1 - P(X < 4) \\ &= 1 - (P(X=0) + P(X=1) + P(X=2) + P(X=3)) \\ &= 1 - \frac{e^{-5} 5^0}{0!} - \frac{e^{-5} 5}{1!} - \frac{e^{-5} 5^2}{2!} - \frac{e^{-5} 5^3}{3!} \\ &= .7350 \end{aligned}$$

$$c) \text{rate per hour} = 5 \Rightarrow \text{rate per 45 minutes} = 0.75(5) = 3.75$$

for Poisson expected value is rate

Problem 8

a)



b)

$$\int_{-\infty}^{\infty} f(y) dy = \int_0^5 \frac{1}{25} y dy + \int_5^{10} \left(\frac{2}{5} - \frac{1}{25} y \right) dy$$

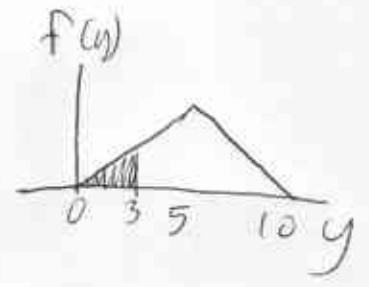
$$= \frac{1}{25} \left[\frac{1}{2} y^2 \right]_0^5 + \left[\frac{2}{5} y - \frac{1}{50} y^2 \right]_5^{10}$$

$$= \frac{25}{50} + \left[4 - 2 - \left(2 - \frac{25}{50} \right) \right]$$

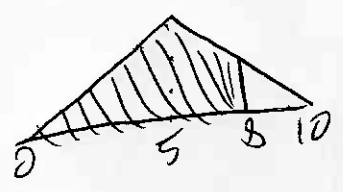
$$= \frac{25}{50} + \frac{25}{50}$$

$$= 1$$

$$\begin{aligned} c) P(Y < 3) &= \int_0^3 \frac{1}{25} y dy = \frac{1}{25} \left[\frac{1}{2} y^2 \right]_0^3 \\ &= \frac{9}{50} \end{aligned}$$



$$d) P(Y \leq 8) = \int_0^5 \frac{1}{25} y dy + \int_5^8 \left(\frac{2}{5} - \frac{1}{25} y \right) dy$$



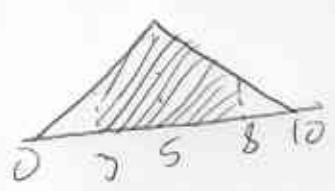
$$= \frac{1}{50} y^2 \Big|_0^5 + \left[\frac{2}{5} y - \frac{1}{50} y^2 \right]_5^8$$

$$= \frac{25}{50} + \left(\frac{16}{5} - \frac{64}{50} - \left(2 - \frac{25}{50} \right) \right)$$

$$= \frac{25}{50} + \left(\frac{160}{50} - \frac{64}{50} - \frac{100}{50} + \frac{25}{50} \right)$$

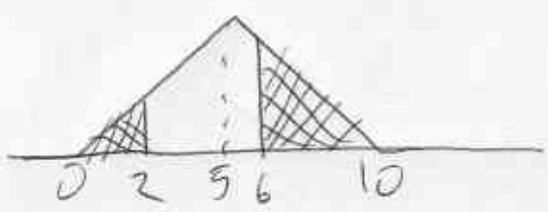
$$= \frac{25}{50} + \frac{21}{50} = \frac{46}{50} = \frac{23}{25}$$

$$e) P(3 < Y < 8) = P(Y < 8) - P(Y < 3)$$



$$= \frac{46}{50} - \frac{9}{50} = \frac{37}{50}$$

$$f) P(X < 2 \text{ or } X > 6) = \int_0^2 \frac{1}{25} y dy + \int_6^{10} \left(\frac{2}{5} - \frac{1}{25} y \right) dy$$



$$= \frac{4}{50} + \left[4 - 2 - \left(\frac{12}{5} - \frac{36}{50} \right) \right]$$

$$= \frac{4}{50} + \left[\frac{100}{50} - \frac{120}{50} + \frac{36}{50} \right] = \frac{20}{50} = \frac{2}{5}$$

Problem 11

a) $P(X \leq 1) = F(1) = \frac{1}{4}$

b) $P(.5 \leq X \leq 1) = F(1) - F(.5)$
 $= \frac{1}{4} - \frac{(1/2)^2}{4}$

$= \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$

c) $P(X > .5) = 1 - P(X < .5)$
 $= 1 - \frac{(1/2)^2}{4} = 1 - \frac{1}{16} = \frac{15}{16}$

d) $\frac{1}{2} = \frac{x_{med}^2}{4} \Rightarrow x_{med} = \sqrt{2}$

e) $F'(x) = \frac{d}{dx} \frac{x^2}{4} \quad 0 \leq x < 2$
 $= \frac{x}{2} \quad 0 \leq x < 2$

$F'(x) = \frac{d}{dx} 0 = 0 \quad x < 0$

$F'(x) = \frac{d}{dx} 1 = 0 \quad x \geq 2$

$$f) E[X] = \int_0^2 x \frac{x}{2} dx$$

$$= \frac{1}{2} \frac{x^3}{3} \Big|_0^2 = \frac{4}{3}$$

$$g) \text{Var}(X) = ?$$

$$E[X^2] = \int_0^2 x^2 \frac{x}{2} dx = \frac{1}{2} \frac{x^4}{4} \Big|_0^2 = 2$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 2 - \left(\frac{4}{3}\right)^2$$

$$= 2 - \frac{16}{9} = \frac{2}{9}$$

$$\text{sd}(X) = \sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3}$$

$$h) E[X^2] = 2$$