Homework 3 Solutions

CH3: 13, 22, 31, 37, 52, 62, 64, 82
CH4: 8, 11

Problem 13

a) $P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$
$= .1 + .15 + .2 + .25$
$= .7$

b) $P(X < 3) = P(X=0) + P(X=1) + P(X=2)$
$= .1 + .15 + .2$
$= .45$

c) $P(X \geq 3) = P(X=3) + P(X=4) + P(X=5) + P(X=6)$
$= .25 + .20 + .06 + .04$
$= .55$

or
$P(X \geq 3) = 1 - P(X < 3) = 1 - .45 = .55$

d) $P(2 \leq X \leq 5) = P(X=2) + P(X=3) + P(X=4) + P(X=5)$
$= .20 + .25 + .20 + .06$
$= .71$

e)

X = "number of lines in use"

$\Rightarrow$ $6 - X$ = "number of lines not in use"

$2 \leq 6 - X \leq 4 \Rightarrow 4 \geq X \geq 2$ $\quad P(2 \leq X \leq 4) = P(X=2) + P(X=3) + P(X=4)$
$= .2 + .25 + .2 = .65$

f) $6 - x \geq 4 \Rightarrow x \leq 2$

$P(X \leq 2) = .45$ (from b)

Problem 22

a) $p(2) = P(X=2) = F(2) - F(1)$

$= .39 - .19$

$= .2$

b) $P(X>3) = 1 - P(X \leq 3)$

$= 1 - F(3)$

$= 1 - .67$

$= .33$

c) $P(2 \leq X \leq 5) = F(5) - F(1)$

$= .97 - .19$

$= .78$

d) $P(2<X<5) = P(3 \leq X \leq 4)$

$= F(4) - F(2)$

$= .92 - .39$

$= .53$

Problem 31

a) $E(X) = \sum x_i P(X=x_i) = 13.5(.2) + 15.9(.5) + 19.1(.3)$

$= 16.38$

$$E(X^2) = \sum_i x_i^2 p(X=x_i) = (13.5)^2(.2) + (15.9)^2(.5) + (19.1)^2(.3)$$
$$= 272.298$$

$$Var(x) = E(X^2) - [E(X)]^2 = 272.298 - (16.38)^2 = 3.9936$$

b) $E[25X - 8.5] = 25E[X] - 8.5$
$$= 401$$

c) $Var(25X - 8.5) = 25^2 Var(x)$
$$= 2496$$

d) $E[h(X)] = E[X - .1X^2] = E[X] - .01E[X^2]$
$$= 16.38 - .01(272.298) = 13.6570$$

(check) $E[X - .1X^2] = \sum_i (x_i - .01x_i^2)\, p(x_i)$
$$= \sum_i x_i p(x_i) - \sum_i .01 x_i^2 p(x_i)$$
$$= E[X] - .01\, E[X^2]$$

Problem 37

$$E\left[\frac{1}{X}\right] = 1\left(\frac{1}{6}\right) + \frac{1}{2}\left(\frac{1}{6}\right) + \frac{1}{3}\left(\frac{1}{6}\right) + \frac{1}{4}\left(\frac{1}{6}\right) + \frac{1}{5}\left(\frac{1}{6}\right) + \frac{1}{6}\,\frac{1}{6}$$
$$= \frac{1}{6} + \frac{1}{12} + \frac{1}{18} + \frac{1}{24} + \frac{1}{30} + \frac{1}{36} = .40833 > 0.2857 = \frac{1}{3.5}$$

Problem 52

X = "number of students in sample who received special accomodations"

$X \sim Bin(25, .02)$

a) $P(X=1) = \binom{25}{1} .02^1 .98^{24} = .3079$

b) $P(X \geq 1) = 1 - P(X<1)$

$= 1 - (P(X=0))$

$= 1 - \left(\binom{25}{0} .02^0 .98^{25}\right)$

$= 1 - .6035 = .3965$

c) $P(X \geq 2) = 1 - P(X \leq 1)$

$= 1 - (P(X=0) + P(X=1))$

$= 1 - .3079 - .6035 = .0886$

d) $E[X] = 25(.02) = .5$

$SD(X) = \sqrt{25(.02)(.98)} = \sqrt{.49} = .7$

$.5 \pm 2(.7) = (-0.9, 1.9)$

So within 2SD of expected number is $X=0$ or $X=1$

$P(X=0) + P(X=1) = 1 - P(X \geq 2) = 1 - .0886 = .9114$

e) "number of hours allowed" $\equiv Y$
for all students

$$Y = 4.5X + 3(25-X)$$
$$= 1.5X + 75$$

$$\Rightarrow E[Y] = E[1.5X + 75]$$
$$= 1.5E[X] + 75$$
$$= 1.5(.5) + 75$$
$$= 75.75$$

$\Rightarrow$ number of hours allowed $\dfrac{75.75}{25} = 3.03$ hours

Problem 62

a) X = "number of reservations who show up"

$X \sim Bin(6, .8)$

At least one individual cannot be accomodated

$\Rightarrow X=5$ or $X=6$ ie $X>4$

$$P(X>4) = P(X=5) + P(X=6)$$
$$= \binom{6}{5} .8^5 .2^1 + \binom{6}{6} .8^6 .2^0 = .6554$$

b) "number of places available" $= 4 - X$

$$E[4-X] = 4 - E[X] = 4 - 6(.8) = -0.8$$

(c) $X =$ "number of passengers on trip"

$$P(X=x) = \sum_{y=3}^{6} P(x \text{ turned up and } y \text{ reserved})$$

$$= \sum_{y=3}^{6} P(x \text{ turned up} \mid y \text{ reserved})\, P(y \text{ reserved})$$

So

$$\begin{aligned} P(X=0) = \;& P(0 \text{ turned up and } 3 \text{ reserved}) \\ &+ P(0 \text{ turned up and } 4 \text{ reserved}) \\ &+ P(0 \text{ turned up and } 5 \text{ reserved}) \\ &+ P(0 \text{ turned up and } 6 \text{ reserved}) \end{aligned}$$

$$= \left(\binom{3}{0} .8^0 .2^3\right)(.1) + \left(\binom{4}{0} .8^0 .2^4\right)(.2) + \left(\binom{5}{0} .8^0 .2^5\right)(.3)$$

$$+ \left(\binom{6}{0} .8^0 .2^6\right)(.4) = .0012$$

$$P(X=1) = \left(\binom{3}{1} .8^1 .2^2\right)(.1) + \left(\binom{4}{1} .8^1 .2^3\right)(.2) + \left(\binom{5}{1} .8^1 .2^4\right)(.3)$$

$$+ \left(\binom{6}{1} .8^1 .2^5\right)(.4)$$

$$= .0173$$

$$P(X=2) = \left(\binom{3}{2} .8^2 .2^1\right)(.1) + \left(\binom{4}{2} .8^2 .2^2\right)(.2) + \left(\binom{5}{2} .8^2 .2^3\right)(.3) + \left(\binom{6}{2} .8^2 .2^4\right)(.4) = .0906$$

$$P(X=3) = \left(\binom{3}{3} .8^3 .2^0\right)(.1) + \left(\binom{4}{3} .8^3 .2\right)(.2) + \left(\binom{5}{3} .8^3 .2^2\right)(.3) + \left(\binom{6}{3} .8^3 .2^3\right)(.4) = .2274$$

$$\begin{aligned} P(X=4) &= 1-\left(P(X=0)+P(X=1)+P(X=2)+P(X=3)\right) \\ &= 1-(.0012+.0173+.0906+.2274) \\ &= .6636 \end{aligned}$$

Problem 64

a) $X \sim Hyp(5, 6, 15)$

samplesize

\# 3 megapixels in population

population size

b) $$P(X=2) = \frac{\binom{6}{2}\binom{9}{3}}{\binom{15}{5}} = \frac{\frac{6!}{2!\,4!}\;\frac{9!}{6!\,3!}}{\frac{15!}{5!\,10!}} = .280$$

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= \frac{\binom{6}{0}\binom{9}{5}}{\binom{15}{5}} + \frac{\binom{6}{1}\binom{9}{4}}{\binom{15}{5}} + \frac{\binom{6}{2}\binom{9}{3}}{\binom{15}{5}} = .573$$

$$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - \left(P(X \leq 2) - P(X=2)\right)$$

$$= 1 - (.573 - .280) = .707$$

c) $E[X] = 5\left(\frac{6}{15}\right) = 2$

$$SD(X) = \sqrt{\left(\frac{15-5}{15-1}\right)5\frac{6}{15}\left(1-\frac{6}{15}\right)} = \sqrt{.857} = .926$$

Problem 82

X = "number of people who arrive in an hour"

$X \sim Poi(5)$

a) $P(X=4) = \frac{e^{-5}5^4}{4!} = .1755$

b) $P(X \geq 4) = 1 - P(X<4)$

$= 1-(P(X=0)+P(X=1)+P(X=2)+P(X=3))$

$= 1 - \frac{e^{-5}5^0}{0!} - \frac{e^{-5}5}{1!} - \frac{e^{-5}5^2}{2!} - \frac{e^{-5}5^3}{3!}$

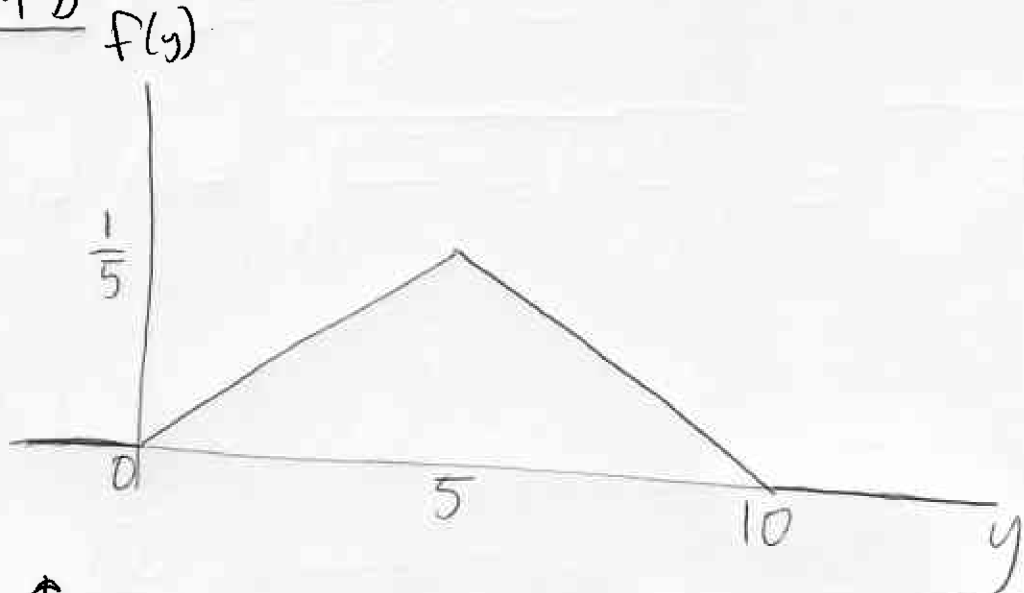
$= .7350$

c) rate per hour $= 5 \Rightarrow$ rate per 45 minutes $= 0.75(5)$
$= 3.75$

for poisson expected value is rate

Problem 8

a)



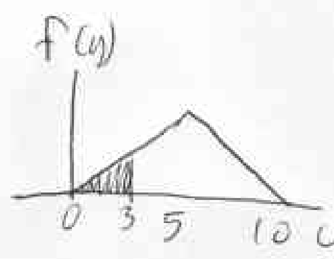
b) $$\int_{-\infty}^{\infty} f(y)\,dy = \int_0^5 \frac{1}{25}y\,dy + \int_5^{10} \frac{2}{5} - \frac{1}{25}y\,dy$$

$$= \frac{1}{25}\left[\frac{1}{2}y^2\right]_0^5 + \left[\frac{2}{5}y - \frac{1}{50}y^2\right]_5^{10}$$

$$= \frac{25}{50} + \left[4 - 2 - \left(2 - \frac{25}{50}\right)\right]$$

$$= \frac{25}{50} + \frac{25}{50}$$

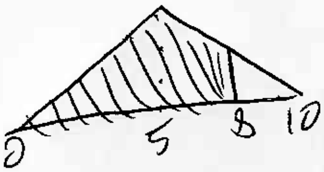
$$= 1$$



c) $$P(Y<3) = \int_0^3 \frac{1}{25}y\,dy = \frac{1}{25}\left[\frac{1}{2}y^2\right]_0^3$$

$$= \frac{9}{50}$$

d) $P(Y \le 8) = \int_0^5 \frac{1}{25} y\,dy + \int_5^8 \frac{2}{5} - \frac{1}{25} y\,dy$



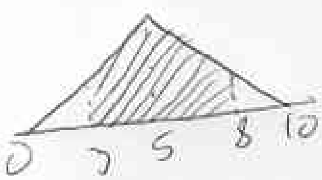
$= \frac{1}{50} y^2 \Big]_0^5 + \frac{2}{5} y - \frac{1}{50} y^2 \Big]_5^8$

$= \frac{25}{50} + \left(\frac{16}{5} - \frac{64}{50} - \left(2 - \frac{25}{50}\right)\right)$

$= \frac{25}{50} + \left(\frac{160}{50} - \frac{64}{50} - \frac{100}{50} + \frac{25}{50}\right)$

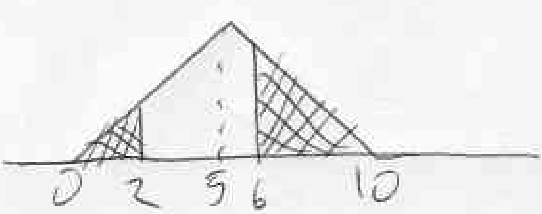
$= \frac{25}{50} + \frac{21}{50} = \frac{46}{50} = \frac{23}{25}$

e) $P(3 < Y < 8) = P(Y < 8) - P(Y < 3)$



$= \frac{46}{50} - \frac{9}{50} = \frac{37}{50}$

f) $P(X < 2 \text{ or } X > 6) = \int_0^2 \frac{1}{25} y\,dy + \int_6^{10} \frac{2}{5} - \frac{1}{25} y\,dy$



$= \frac{4}{50} + \left[4 - 2 - \left(\frac{12}{5} - \frac{36}{50}\right)\right]$

$= \frac{4}{50} + \left[\frac{100}{50} - \frac{120}{50} + \frac{36}{50}\right] = \frac{20}{50} = \frac{2}{5}$

Problem 11

a) $P(X \leq 1) = F(1) = \frac{1}{4}$

b) $P(.5 \leq X \leq 1) = F(1) - F(.5)$

$= \frac{1}{4} - \frac{(1/2)^2}{4}$

$= \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$

c) $P(X > .5) = 1 - P(X < .5)$

$= 1 - \frac{(1/2)^2}{4} = 1 - \frac{1}{16} = \frac{15}{16}$

d) $\frac{1}{2} = \frac{x_{med}^2}{4} \Rightarrow x_{med} = \sqrt{2}$

e) $F'(x) = \frac{d}{dx} \frac{x^2}{4} \qquad 0 \leq x < 2$

$= \frac{x}{2} \qquad 0 \leq x < 2$

$F'(x) = \frac{d}{dx} 0 = 0 \qquad x < 0$

$F'(x) = \frac{d}{dx} 1 = 0 \qquad x \geq 2$

f) $E[X] = \int_0^2 x \frac{x}{2}\,dx$

$= \frac{1}{2}\frac{x^3}{3}\Big|_0^2 = \frac{4}{3}$

g) $Var(X) = ?$

$E[X^2] = \int_0^2 x^2 \frac{x}{2}\,dx = \frac{1}{2}\frac{x^4}{4}\Big|_0^2 = 2$

$Var(X) = E(X^2) - [E(X)]^2$

$= 2 - \left(\frac{4}{3}\right)^2$

$= 2 - \frac{16}{9} = \frac{2}{9}$

$sd(X) = \sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3}$

h) $E[X^2] = 2$