Homework #6 Solutions

Chapter 6 Problem 10

a. $E(\overline{X}^2) = Var(\overline{X}) + [E(\overline{X})]^2 = \frac{s^2}{n} + m^2$, so the bias of the estimator $\overline{X}^2$ is $\frac{s^2}{n}$; thus $\overline{X}^2$ tends to overestimate m^2.

b. $E(\overline{X}^2 - kS^2) = E(\overline{X}^2) - kE(S^2) = m^2 + \frac{s^2}{n} - ks^2$, so with $k = \frac{1}{n}$, $E(\overline{X}^2 - kS^2) = m^2$.

Problem 15

a. $E(X^2) = 2q$ implies that $E\left(\frac{X^2}{2}\right) = q$. Consider $\hat{q} = \frac{\sum X_i^2}{2n}$. Then $E(\hat{q}) = E\left(\frac{\sum X_i^2}{2n}\right) = \frac{\sum E(X_i^2)}{2n} = \frac{\sum 2q}{2n} = \frac{2nq}{2n} = q$, implying that $\hat{q}$ is an unbiased estimator for q.

b. $\sum x_i^2 = 1490.1058$, so $\hat{q} = \frac{1490.1058}{20} = 74.505$

Problem 22

a. $E(X) = \int_0^1 x(q+1)x^q\,dx = \frac{q+1}{q+2} = 1 - \frac{1}{q+2}$, so the moment estimator $\hat{q}$ is the solution to $\overline{X} = 1 - \frac{1}{\hat{q}+2}$, yielding $\hat{q} = \frac{1}{1-\overline{X}} - 2$. Since $\overline{x} = .80, \hat{q} = 5 - 2 = 3.$

b. $f(x_1, \ldots, x_n; q) = (q+1)^n (x_1 x_2 \ldots x_n)^q$, so the log likelihood is $n\ln(q+1) + q\sum \ln(x_i)$. Taking $\frac{d}{dq}$ and equating to 0 yields $\frac{n}{q+1} = -\sum \ln(x_i)$, so $\hat{q} = -\frac{n}{\sum \ln(X_i)} - 1$. Taking $\ln(x_i)$ for each given x_i yields ultimately $\hat{q} = 3.12$.

Problem 28

a. $\left(\frac{x_1}{q}\exp\left[-x_1^2/2q\right]\right)\ldots\left(\frac{x_n}{q}\exp\left[-x_n^2/2q\right]\right)=\left(x_1\ldots x_n\right)\frac{\exp\left[-\Sigma x_i^2/2q\right]}{q^n}$. The natural log of the likelihood function is $\ln\left(x_i\ldots x_n\right)-n\ln\left(q\right)-\frac{\Sigma x_i^2}{2q}$. Taking the derivative wrt q and equating to 0 gives $-\frac{n}{q}+\frac{\Sigma x_i^2}{2q^2}=0$, so $nq=\frac{\Sigma x_i^2}{2}$ and $q=\frac{\Sigma x_i^2}{2n}$. The mle is therefore $\hat{q}=\frac{\Sigma X_i^2}{2n}$, which is identical to the unbiased estimator suggested in Exercise 15.

b. For x > 0 the cdf of X if $F\left(x;q\right)=P\left(X\le x\right)$ is equal to $1-\exp\left[\frac{-x^2}{2q}\right]$. Equating this to .5 and solving for x gives the median in terms of q: $.5=\exp\left[\frac{-x^2}{2q}\right]$ implies that $\ln(.5)=\frac{-x^2}{2q}$, so $x=\tilde{m}=\sqrt{1.38630}$. The mle of $\tilde{m}$ is therefore $\left(1.38630\hat{q}\right)^{\frac{1}{2}}$.

Chapter 7 Problem 1

a. $z_{\alpha/2}=2.81$ implies that $\alpha/2=1-\Phi(2.81)=.0025$, so $\alpha=.005$ and the confidence level is $100(1-\alpha)\%=99.5\%$.

b. $z_{\alpha/2}=1.44$ for $\alpha=2\left[1-\Phi(1.44)\right]=.15$, and $100(1-\alpha)\%=85\%$.

c. 99.7% implies that $\alpha=.003$, $\alpha/2=.0015$, and $z_{.0015}=2.96$. (Look for cumulative area .9985 in the main body of table A.3, the Z table.)

d. 75% implies $\alpha=.25$, $\alpha/2=.125$, and $z_{.125}=1.15$.

Problem 4

a. $58.3\pm\frac{1.96(3)}{\sqrt{25}}=58.3\pm1.18=(57.1,59.5)$

b. $58.3\pm\frac{1.96(3)}{\sqrt{100}}=58.3\pm.59=(57.7,58.9)$

c. $58.3 \pm \frac{2.58(3)}{\sqrt{100}} = 58.3 \pm .77 = (57.5, 59.1)$

d. 82% confidence $\Rightarrow 1-\alpha = .82 \Rightarrow \alpha = .18 \Rightarrow \alpha/2 = .09$, so $z_{\alpha/2} = z_{.09} = 1.34$ and the interval is $58.3 \pm \frac{1.34(3)}{\sqrt{100}} = (57.9, 58.7)$.

e. $n = \left[\frac{2(2.58)3}{1}\right]^2 = 239.62$ so n = 240.

Problem 14

a. $89.10 \pm 1.96\frac{3.73}{\sqrt{169}} = 89.10 \pm .56 = (88.54, .89.66)$. Yes, this is a very narrow interval. It appears quite precise.

b. $n = \left[\frac{(1.96)(.16)}{.5}\right]^2 = 245.86 \Rightarrow n = 246$.

Problem 23

a. $\hat{p} = \frac{24}{37} = .6486$; The 99% confidence interval for p is

$$\frac{.6486 + \frac{(2.58)^2}{2(37)} \pm 2.58\sqrt{\frac{(.6486)(.3514)}{37} + \frac{(2.58)^2}{4(37)^2}}}{1 + \frac{(2.58)^2}{37}} = \frac{.7386 \pm .2216}{1.1799} = (.438, .814)$$

b. $$n = \frac{2(2.58)^2(.25) - (2.58)^2(.01) \pm \sqrt{4(2.58)^4(.25)(.25-.01) + .01(2.58)^4}}{.01}$$
$$= \frac{3.261636 \pm 3.3282}{.01} \approx 659$$

Problem 25

a. $$n = \frac{2(1.96)^2(.25) - (1.96)^2(.01) \pm \sqrt{4(1.96)^4(.25)(.25-.01) + .01(1.96)^4}}{.01} \approx 381$$

b. $n = \dfrac{2(1.96)^2\left(\frac{1}{3}\cdot\frac{2}{3}\right) - (1.96)^2(.01) \pm \sqrt{4(1.96)^4\left(\frac{1}{3}\cdot\frac{2}{3}\right)\left(\frac{1}{3}\cdot\frac{2}{3} - .01\right) + .01(1.96)^4}}{.01} \approx 339$

Problem 34

n = 14, $\bar{x} = 8.48$, s = .79; $t_{.05,13} = 1.771$

a. A 95% lower confidence bound: $8.48 - 1.771\left(\dfrac{.79}{\sqrt{14}}\right) = 8.48 - .37 = 8.11$. With 95% confidence, the value of the true mean proportional limit stress of all such joints lies in the interval $(8.11, \infty)$. If this interval is calculated for sample after sample, in the long run 95% of these intervals will include the true mean proportional limit stress of all such joints. We must assume that the sample observations were taken from a normally distributed population.

b. A 95% lower prediction bound: $8.48 - 1.771(.79)\sqrt{1 + \dfrac{1}{14}} = 8.48 - 1.45 = 7.03$. If this bound is calculated for sample after sample, in the long run 95% of these bounds will provide a lower bound for the corresponding future values of the proportional limit stress of a single joint of this type.