NAME: Model Solutions

Math 324 FALL 2004: Section 2 MWF 10-11
Midterm 1 **Date: Oct 15, 2004**

Instructions: Answer all questions. Show working where it is useful to do so. You have 50 minutes. To allow others to fully concentrate at the end please do not leave in the last 10 minutes. You should submit your page of notes with your test paper.

Question 1. (*25 points*)

A shipment of computer motherboards contains 42 good ones and 8 defective ones. An inspector selects 5 devices at random without replacement. The inspector will approve the shipment if there are 2 or fewer defectives in the sample.

(a) What is the probability that exactly three are good?

Let X = "number of good motherboards in sample" $X \sim Hyp(5, 42, 50)$

$$P(X=3) = \frac{\binom{42}{3}\binom{8}{2}}{\binom{50}{5}} = 0.1517 \ (4dp)$$

$\boxed{10pts}$

(b) What is the probability that the inspector approves the shipment?

Approve shipment implies $X=3$ or 4 or 5

$$P(X=4) = \frac{\binom{42}{4}\binom{8}{1}}{\binom{50}{5}} = \frac{\binom{42}{4}}{\binom{50}{5}} 8 = .4226 \ (4dp)$$

$\boxed{15pts}$

$$P(X=5) = \frac{\binom{42}{5}\binom{8}{0}}{\binom{50}{5}} = \frac{\binom{42}{5}}{\binom{50}{5}} = .4015 \ (4dp)$$

$$P(\text{Approves}) = P(X=3) + P(X=4) + P(X=5)$$
$$= .1517 + .4226_1 + .4015 = .9758$$

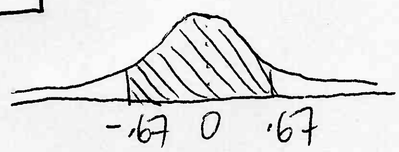
Question 2. (*25 points*)

Suppose $X \sim N(100, 15)$ and $Y \sim Unif(0, 1)$ and X and Y are completely independent.

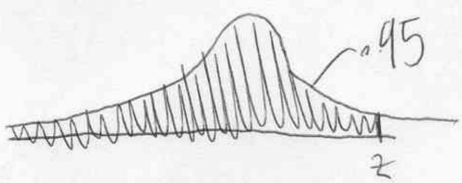
(a) What is the probability that X is between 90 and 110?

$\boxed{6\text{pts}}$

$$P(90<X<110) = P\left(\frac{90-100}{15} < \frac{X-100}{15} < \frac{110-100}{15}\right)$$
$$= P(-.67 < Z < .67)$$
$$= \Phi(.67) - \Phi(-.67)$$
$$= .7486 - .2514 = .4972$$



(b) What value of X is the 95 percentile?



$\Phi(z) = .95$ from the table closest are $\Phi(1.64) = .9495$ and $\Phi(1.65) = .9505$ so take $z = 1.645$

$\boxed{6\text{pts}}$

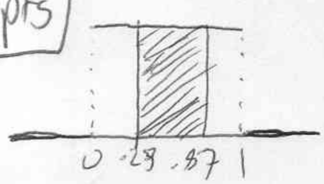
Unstandardize $x = (1.645)(15) + 100 = 124.675$

(c) What is the probability that X is between 90 and 110 and Y is between 0.23 and 0.87?

$\boxed{6\text{pts}}$

Since X, Y are independent

$$P((90<X<110) \cap (.23<Y<.87))$$
$$= P(90<X<110)\,P(.23<Y<.87)$$
$$= (.4972)(.64)$$
$$= .3182$$



$$P(.23<Y<.87) = (.87-.23)(1)$$
$$= .64$$

(d) What is the probability that X is between 90 and 110 or Y is between 0.23 and 0.87 or both conditions hold true?

$\boxed{7\text{pts}}$

$$P((90<X<110) \cup (.23<Y<.87)) = P(90<X<110) + P(.23<Y<.87) - P((90<X<110) \cap (.23<Y<.87))$$
$$= .4972 + .64 - .3182$$
$$= .8190$$

Question 3. (*25 points*)

An email mailbox receives messages at a rate of 15 messages per hour. 75% of all email messages can be considered spam. Suppose that the mailbox is scanned by a perfect spam scanner, that is it never makes a mistake in classifying a message as spam or not spam. You should assume that email arrives at random intervals.

(a) What is the probability that 15 email messages arrive in a randomly chosen hour and all 15 are spam messages?

$$P(15 \text{ arrive and } 15 \text{ are spam}) = P(15 \text{ are spam} \mid 15 \text{ arrive}) P(15 \text{ arrive})$$

$$= \binom{15}{15} .75^{15} .25^{0} \frac{e^{-15} 15^{15}}{15!}$$

8pts

$$= .0014 \text{ (4dp)}$$

(b) What is the probability that y arrive ($y \geq 15$) in the hour and 15 of these are spam messages?

$$P(y \text{ arrive and } 15 \text{ are spam}) = P(15 \text{ are spam} \mid y \text{ arrive}) P(y \text{ arrive})$$

$$= \binom{y}{15} .75^{15} .25^{y-15} \frac{e^{-15} 15^{y}}{y!}$$

9pts

(c) What is the probability of getting 15 spam emails in a randomly chosen hour?

Spam emails are at a rate of $15(.75) = 11.25$ per hour

so probability is

$$\frac{e^{-11.25} 11.25^{15}}{15!} = .0582$$

8pts

Question 4. (*25 points*)

Suppose that Y is a random variable that has the following pdf

$$f(y) = \begin{cases} \frac{1}{9}(4-y^2) & -1 \le y \le 2 \\ 0 & \text{Otherwise} \end{cases}$$

(a) Compute the cdf.

$$F(y) = \int_{-1}^{y} \frac{1}{9}(4-x^2)\,dx \qquad \text{For } -1 \le y \le 2$$

$$= \frac{4}{9}x - \frac{x^3}{27}\Big|_{-1}^{y} = \frac{4}{9}y - \frac{y^3}{27} - \left(\frac{-4}{9} + \frac{1}{27}\right)$$

6pts

$$= \frac{4}{9}y - \frac{y^3}{27} + \frac{11}{27}$$

$F(y) = 0 \quad y < -1$

$F(y) = 1 \quad y > 2$

(b) What is the median of this distribution?

Median is value y_p which solves

$$.5 = F(y_p) = \frac{4}{9}y_p - \frac{y_p^2}{27} + \frac{11}{27}$$

6pts

Rearranging and multiplying through by 27 gives

$$y_p^3 - 12y_p + 2.5 = 0$$

So y_p is a solution of above equation.

(Bonus: roots are $-3.5639, 0.2091, 3.3548$)

$.2091$ is median.

(c) What is the expected value of Y?

$$E[Y] = \int_{-1}^{2} y \frac{1}{9}(4-y^2)\,dy$$

$\boxed{6\text{pts}}$

$$= \frac{1}{9}\left[\frac{4y^2}{2} - \frac{y^4}{4}\right]_{-1}^{2} = \frac{1}{9}\left[\left(\frac{16}{2} - \frac{16}{4}\right) - \left(\frac{4}{2} - \frac{1}{4}\right)\right]$$

$$= \frac{1}{9}\left[8 - 4 - 2 + \frac{1}{4}\right] = \frac{1}{9}\frac{9}{4} = \frac{1}{4}$$

(d) What is the variance of Y?

$$Var(Y) = E[Y^2] - (E[Y])^2$$

$$E[Y^2] = \int_{-1}^{2} y^2 \frac{1}{9}(4-y^2)\,dy$$

$\boxed{7\text{pts}}$

$$= \frac{1}{9}\left[\frac{4y^3}{3} - \frac{y^5}{5}\right]_{-1}^{2}$$

$$= \frac{1}{9}\left[\frac{2^5}{3} - \frac{2^5}{5} - \left(\frac{-4}{3} + \frac{1}{5}\right)\right]$$

$$= \frac{1}{9}\left[\frac{(5-3)2^5}{15} + \frac{17}{15}\right]$$

$$= \frac{1}{9}\left[\frac{64}{15} + \frac{17}{15}\right] = \frac{1}{9}\frac{81}{15} = \frac{9}{15}$$

$$Var(Y) = \frac{9}{15} - \left(\frac{1}{4}\right)^2 = \frac{9}{15} - \frac{1}{16} = \frac{144-15}{240} = \frac{129}{240}\ (= .5375)$$